

Theoretical Question 2: Ray tracing and generation of entangled light

Part A. Light propagation in isotropic dielectric media

A.1 0.4 pt

Ans: $\frac{1}{\sqrt{\mu_0\epsilon}}$

Solution:

From $\vec{k} \times \vec{E} = \omega \vec{B} = \omega \mu_0 \vec{H}$ and $\vec{k} \times \vec{H} = -\omega \vec{D}$, one obtains $\vec{k} \times (\vec{k} \times \vec{E}) = -\omega^2 \mu_0 \vec{D}$. By using the given identity $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$, one finds $\vec{k} \times (\vec{k} \times \vec{E}) = \vec{k}(\vec{k} \cdot \vec{E}) - k^2 \vec{E}$. Since $\vec{D} \cdot \vec{k} = 0$ and $\vec{D} = \epsilon \vec{E}$, we find $\vec{k} \times (\vec{k} \times \vec{E}) = -k^2 \vec{E}$ and the relation $\vec{k} \times (\vec{k} \times \vec{E}) = -\omega^2 \mu_0 \vec{D}$ reduces to $-k^2 \vec{E} = -\omega^2 \mu_0 \epsilon \vec{E}$.

Now the phase velocity is determined by $\frac{d(\vec{k} \cdot \vec{r} - \omega t)}{dt} = 0$, we find that the phase velocity $\vec{v}_p = \frac{d\vec{r}}{dt} = \frac{\omega}{k} \hat{k}$. Clearly, we have $\frac{\omega}{k} = \frac{1}{\sqrt{\mu_0\epsilon}}$. Hence $v_p = \frac{1}{\sqrt{\mu_0\epsilon}}$.

A.2 0.2 pt

Ans: $c\sqrt{\mu_0\epsilon}$

Solution:

From $v_p = \frac{1}{\sqrt{\mu_0\epsilon}} = \frac{c}{n}$, we find $n = c\sqrt{\mu_0\epsilon}$

A.3 0.4 pt

Ans: \hat{k} , $v_r = v_p = \frac{1}{\sqrt{\mu_0\epsilon}}$

Solution:

To find the speed of the ray, we first note that the direction of the energy flow, given by the Poynting vector $\vec{S} = \vec{E} \times \vec{H}$, is in the same direction of \vec{k} . The electromagnetic energy density $u = u_e + u_m$ with $u_e = \frac{1}{2} \vec{E} \cdot \vec{D}$ and $u_m = \frac{1}{2} \vec{B} \cdot \vec{H}$.

Now, from $\vec{k} \times \vec{H} = -\omega \vec{D}$, one has $\vec{D} = -\frac{1}{v_p} \hat{k} \times \vec{H}$. Hence $u_e = -\frac{1}{2v_p} \vec{E} \cdot \hat{k} \times \vec{H} = \frac{1}{2v_p} \hat{k} \cdot \vec{E} \times \vec{H}$. Similarly, from $\vec{k} \times \vec{E} = \omega \vec{B}$, we find $u_m = \frac{1}{2v_p} \vec{B} \cdot \hat{k} \times \vec{E} = \frac{1}{2v_p} \hat{k} \cdot \vec{E} \times \vec{B}$. Hence $u = \frac{1}{v_p} \hat{k} \cdot \vec{E} \times \vec{B}$.

We find $v_r = S/u = v_p = \frac{1}{\sqrt{\mu_0\epsilon}}$.

Part B. Light propagation in uniaxial dielectric media

B.1 1.5pt

Ans: $n = n_o$, $\hat{B} = \pm \hat{k} \times \hat{y} = \pm(-\cos\theta, 0, \sin\theta)$, $\hat{D} = \pm \hat{y}$ or $n = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2\theta + n_e^2 \cos^2\theta}}$, $\hat{B} = \pm \hat{y}$, $\hat{D} = \pm \hat{y} \times \hat{k} = \pm(\cos\theta, 0, -\sin\theta)$. For $\theta = 0$, there is only one permitted value for the refractive index

Solution:

From $\vec{k} \times \vec{E} = \omega \mu_0 \vec{H}$ and $\vec{k} \times \vec{H} = -\omega \vec{D}$, one obtains $\vec{k} \times (\vec{k} \times \vec{E}) = -\omega^2 \mu_0 \vec{D}$. Writing out

components and using $\omega = \frac{c}{n}k$, we find

$$\begin{aligned} -\cos^2 \theta E_x + \cos \theta \sin \theta E_z &= -\frac{n_o^2}{n^2} E_x, \\ -\cos^2 \theta E_y - \sin^2 \theta E_y &= -\frac{n_o^2}{n^2} E_y, \\ -\sin^2 \theta E_z + \cos \theta \sin \theta E_x &= -\frac{n_e^2}{n^2} E_z. \end{aligned}$$

After a bit rearrangement, we obtain

$$\begin{aligned} \left(1 - \frac{n_o^2}{n^2}\right) E_y &= 0 \\ \left(\frac{n_o^2}{n^2} - \cos^2 \theta\right) E_x + \cos \theta \sin \theta E_z &= 0 \\ \cos \theta \sin \theta E_x + \left(\frac{n_o^2}{n^2} - \sin^2 \theta\right) E_z &= 0. \end{aligned}$$

The vanishing of the determinant yields

$$\left(1 - \frac{n_o^2}{n^2}\right) \left[\left(\frac{n_o^2}{n^2} - \cos^2 \theta\right)\left(\frac{n_e^2}{n^2} - \sin^2 \theta\right) - \sin^2 \theta \cos^2 \theta\right] = 0. \quad (1)$$

Clearly, for a general θ , we have two solutions for n :

$$(1) \quad n = n_o$$

In this case, $E_x = E_z = 0$. \vec{E} is parallel to the y axis. From $\vec{k} \times \vec{E} = \omega \vec{B}$ and $\vec{k} \times (\mu_0 \vec{B}) = -\omega \vec{D}$, we obtain the directions of \vec{B} and \vec{D} as $\hat{B} = \pm \hat{k} \times \hat{y} = \pm(-\cos \theta, 0, \sin \theta)$ and $\hat{D} = -\hat{k} \times \hat{B} = \pm(0, 1, 0) = \pm \hat{y}$.

$$(2) \quad \left(\frac{n_o^2}{n^2} - \cos^2 \theta\right)\left(\frac{n_e^2}{n^2} - \sin^2 \theta\right) - \sin^2 \theta \cos^2 \theta = 0.$$

After rearrangement, we find $n = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2 \theta + n_e^2 \cos^2 \theta}}$. Clearly, at $\theta = 0$, $n = n_o$, there is only one refractive index. This is the direction of the optic axis.

In this case, $E_y = 0$. Hence \vec{E} lies in the xz plane. Hence the relation $\vec{k} \times \vec{E} = \omega \vec{B}$ implies $\hat{B} = \pm \hat{y}$. The relation $\vec{k} \times (\mu_0 \vec{B}) = -\omega \vec{D}$ implies $\hat{D} = \pm \hat{y} \times \hat{k}$.

B.2 0.8 pt

Ans: (1) when $n = n_o$, $\hat{E} = \pm \hat{y}$ and this is an ordinary ray. $\tan \alpha = 0$.

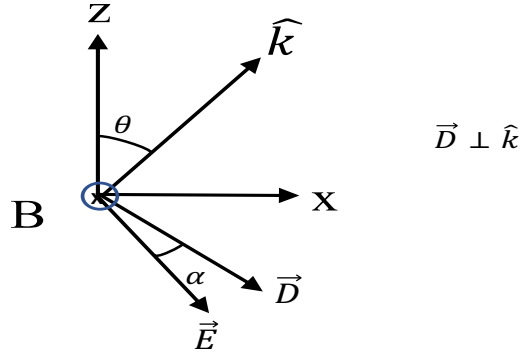
(2) when $n = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2 \theta + n_e^2 \cos^2 \theta}}$, $\hat{E} = \pm \frac{1}{\sqrt{n_e^4 \cos^2 \theta + n_o^4 \sin^2 \theta}}(-n_e^2 \cos \theta, 0, n_o^2 \sin \theta)$ and this is an extraordinary ray. $\tan \alpha = \frac{(n_o^2 - n_e^2) \tan \theta}{n_e^2 + n_o^2 \tan^2 \theta}$.

Solution:

(1) For $n = n_o$, both \vec{E} and \vec{D} are parallel to the y axis. This is an ordinary ray with $\tan \alpha = 0$.

(2) For $n = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2 \theta + n_e^2 \cos^2 \theta}}$, $n \neq n_o$, $E_y = 0$. By substituting n back into the equations of E_x and E_z , we find that $\frac{n_o^2}{n_e^2} \sin \theta E_x + \cos \theta E_z = 0$. Hence the electric field lies in xz plane with $\hat{E} = \pm \frac{1}{\sqrt{n_e^4 \cos^2 \theta + n_o^4 \sin^2 \theta}} (-n_e^2 \cos \theta, 0, n_o^2 \sin \theta)$ (\vec{B} points in $\mp y$ direction.). Therefore, \vec{E} is not perpendicular to \vec{k} and lies in the xz plan in together with \vec{D} and \vec{k} . This is the extraordinary ray.

Since $\vec{k} \times \vec{H} = -\omega \vec{D}$, \vec{D} is perpendicular to \hat{k} . Hence $\hat{D} = \pm(-\cos \theta, 0, \sin \theta)$. Let $\vec{B} = \hat{y}$, the relative orientation of \vec{E} and \vec{D} for a given θ are shown in the following figure for the case when $n_e < n_o$.



Let the angle relative to x axis be θ_1 and θ_2 for \vec{E} and \vec{D} . We have $\tan \theta_2 = -\tan \theta$ and $\tan \theta_1 = -\frac{n_o^2}{n_e^2} \tan \theta$. Hence $\tan \alpha = \tan(\theta_2 - \theta_1) = \frac{\tan \theta_2 - \tan \theta_1}{1 + \tan \theta_1 \tan \theta_2} = \frac{(n_o^2 - n_e^2) \tan \theta}{n_e^2 + n_o^2 \tan^2 \theta}$. The same result remains when $n_e > n_o$ except that $\tan \alpha < 0$, indicating that the relative orientation of \vec{E} and \vec{D} is reversed.

B.3 0.6 pt

Ans: $n = n_o$, $\vec{E} = \pm \hat{k} \times \hat{z} / \sin \theta$ and this is an ordinary ray.

when $n = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2 \theta + n_e^2 \cos^2 \theta}}$, $\hat{E} = \pm \frac{1}{\sqrt{n_e^4 \cos^2 \theta + n_o^4 \sin^2 \theta}} \frac{-n_e^2 \cos \theta \hat{k} + (n_o^2 \sin^2 \theta - n_e^2 \cos^2 \theta) \hat{z}}{\sin \theta}$ and this is an extraordinary ray.

Solution: The problem has an axial symmetry so that in the plane formed by the z axis and \hat{k} , one can write $\vec{k} = k_z \hat{z} + k_\perp \hat{k}_\perp$ and $\vec{E} = E_z \hat{z} + E_\perp \hat{k}_\perp$, where \hat{k}_\perp is perpendicular to \hat{z} . Clearly, we $k_z = k \cos \theta$, $k_\perp = k \sin \theta$, $E_z = E \cos \theta$, and $E_\perp = E \sin \theta$. Writing out the components for the equation: $\vec{k} \times (\vec{k} \times \vec{E}) = -\omega^2 \mu_0 \vec{D}$, we get exactly the same equations except that E_x is replaced by E_\perp . Hence all of the solutions are the same except \hat{x} is replaced by \hat{k}_\perp . Since $\hat{k}_\perp \sin \theta = \hat{k} - \cos \theta \hat{z}$, we obtain that when $n = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2 \theta + n_e^2 \cos^2 \theta}}$, $\hat{E} = \pm \frac{1}{\sqrt{n_e^4 \cos^2 \theta + n_o^4 \sin^2 \theta}} [-n_e^2 \cos \theta \frac{(\hat{k} - \cos \theta \hat{z})}{\sin \theta} + n_o^2 \sin \theta \hat{z}] =$

$$\pm \frac{1}{\sqrt{n_e^4 \cos^2 \theta + n_o^4 \sin^2 \theta}} \frac{-n_e^2 \cos \theta \hat{k} + (n_o^2 \sin^2 \theta - n_e^2 \cos^2 \theta) \hat{z}}{\sin \theta}.$$

B.4 0.8 pt

Ans: (1) $n = n_o$, $\tan \alpha_r = 0$, $v_r = \frac{c}{n_o}$, $\hat{S} = (\sin \theta, 0, \cos \theta)$

(2) $n = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2 \theta + n_e^2 \cos^2 \theta}}$, $\tan \alpha_r = \frac{(n_o^2 - n_e^2) \tan \theta}{n_e^2 + n_o^2 \tan^2 \theta}$, $v_r = \frac{c}{n_o n_e} \sqrt{\frac{n_e^4 \cos^2 \theta + n_o^4 \sin^2 \theta}{n_e^2 \cos^2 \theta + n_o^2 \sin^2 \theta}}$

$\hat{S} = \frac{1}{\sqrt{n_e^4 \cos^2 \theta + n_o^4 \sin^2 \theta}} (n_o^2 \sin \theta, 0, n_e^2 \cos \theta)$

(3) $n_s = \sqrt{(\hat{S} \cdot \hat{x})^2 n_e^2 + (\hat{S} \cdot \hat{z})^2 n_o^2}$

Solution:

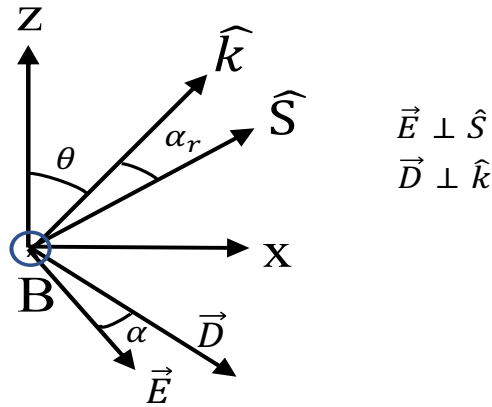
The direction of the energy flow is given by the Poynting vector, $\vec{S} = \vec{E} \times \vec{H}$. Let the energy density of EM wave be u and the ray velocity be v_r . Then $v_r = \frac{S}{u}$. Here $u = u_e + u_m$ with $u_e = \frac{1}{2} \vec{E} \cdot \vec{D}$ and $u_m = \frac{1}{2} \vec{B} \cdot \vec{H}$. There are two cases:

(i) $n = n_o$, $\vec{E} = (0, E, 0)$, $\vec{D} = \epsilon \vec{E}$, $\vec{k} \times \vec{E} = \omega \mu_0 \vec{H}$, $\vec{k} \times \vec{H} = -\omega \vec{D}$.

\hat{k} , \vec{E} and \vec{H} are mutually perpendicular to each other. Hence \vec{S} is parallel to \hat{k} , i.e., $\hat{S} = (\sin \theta, 0, \cos \theta)$ and $\tan \alpha_r = 0$.

Now from $\vec{k} \times \vec{H} = -\omega \vec{D}$, one has $\vec{D} = -\frac{1}{\omega} \hat{k} \times \vec{H}$. Hence $u_e = -\frac{1}{2\omega} \vec{E} \cdot \hat{k} \times \vec{H} = \frac{1}{2\omega} \hat{k} \cdot \vec{E} \times \vec{H}$. Similarly, we find $u_m = \frac{1}{2\omega} \vec{H} \cdot \hat{k} \times \vec{E} = \frac{1}{2\omega} \hat{k} \cdot \vec{E} \times \vec{H}$. Hence $u = \frac{1}{\omega} \hat{k} \cdot \vec{E} \times \vec{H}$. Since $\hat{S} = \hat{k}$, we find $u = \frac{S}{\omega}$. Hence $v_r = \frac{S}{u} = v_p = \frac{\omega}{k} = \frac{c}{n_o}$.

(ii) $n = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2 \theta + n_e^2 \cos^2 \theta}}$. In this case, we can take $\vec{B} = (0, B, 0)$ (negative y direction works as well). \vec{D} , \vec{E} and \hat{k} are in the xz plane and \vec{D} is perpendicular to \hat{k} . Therefore, the angle between $\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$ and \hat{k} is equal to the angle between \vec{D} and \vec{E} , i.e., $\alpha_r = \alpha$. This is shown in the following figure when $n_e < n_o$ (for $n_e > n_o$, both α and α_r are negative, the relative orientation of \vec{E} and \vec{D} is reversed and ordering of \hat{S} and \hat{k} are switched).



Therefore, from problem (d) (ii), we get $\tan \alpha_r = \tan \alpha = \frac{(n_o^2 - n_e^2) \tan \theta}{n_e^2 + n_o^2 \tan^2 \theta}$. Now, because $u = \frac{1}{v_p} \hat{k} \cdot \vec{E} \times \vec{H} = \frac{1}{v_p} |\vec{E} \times \vec{H}| \cos \alpha$, we obtain $v_r = \frac{S}{u} = \frac{v_p}{\cos \alpha}$. Hence the phase speed v_p and the ray speed are related by $v_p = v_r \cos \alpha$. From $\tan \alpha$, one finds $\cos \alpha = \frac{n_e^2 \cos^2 \theta + n_o^2 \sin^2 \theta}{\sqrt{n_e^4 \cos^2 \theta + n_o^4 \sin^2 \theta}}$.

Hence $v_r = \frac{c}{n \cos \alpha} = \frac{c}{n_o n_e} \sqrt{\frac{n_e^4 \cos^2 \theta + n_o^4 \sin^2 \theta}{n_e^2 \cos^2 \theta + n_o^2 \sin^2 \theta}}$.

Clearly, $\hat{S} = (\sin(\theta + \alpha), \cos(\theta + \alpha))$. Since $\sin \alpha = \frac{(n_o^2 - n_e^2) \sin \theta \cos \theta}{\sqrt{n_e^4 \cos^2 \theta + n_o^4 \sin^2 \theta}}$ and $\cos \alpha = \frac{n_e^2 \cos^2 \theta + n_o^2 \sin^2 \theta}{\sqrt{n_e^4 \cos^2 \theta + n_o^4 \sin^2 \theta}}$, we find $\hat{S} = \frac{1}{\sqrt{n_e^4 \cos^2 \theta + n_o^4 \sin^2 \theta}} (n_o^2 \sin \theta, 0, n_e^2 \cos \theta)$.

From $n_s^2 = \left(\frac{c}{v_r}\right)^2 = n_o^2 n_e^2 \frac{n_e^2 \cos^2 \theta + n_o^2 \sin^2 \theta}{n_e^4 \cos^2 \theta + n_o^4 \sin^2 \theta} = \frac{(n_o^2 \sin \theta)^2 n_e^2 + (n_e^2 \cos \theta)^2 n_o^2}{n_e^4 \cos^2 \theta + n_o^4 \sin^2 \theta}$, we find $n_s = (\hat{S} \cdot \hat{x})^2 n_e^2 + (\hat{S} \cdot \hat{z})^2 n_o^2$.

B.5 1.1 pt

Ans: $\bar{A} = P_1(n^2 \sin^2 \theta_1 - P_1)$, $\bar{B} = -2P_3(n^2 \sin^2 \theta_1 - P_1)$, $\bar{C} = P_2 n^2 \sin^2 \theta_1 - P_3^2$.

$$\phi = 0, \tan \theta_2 = \frac{nn_e \sin \theta_1}{n_o \sqrt{n_o^2 - n^2 \sin^2 \theta_1}}.$$

$$\phi = \pi/2, \tan \theta_2 = \frac{nn_o \sin \theta_1}{n_e \sqrt{n_e^2 - n^2 \sin^2 \theta_1}}.$$

Solution:

Let the distance along z axis between A and B be d and the point of the interface that the ray passes be the origin O . The coordinates of B and A points can be expressed as $(h_2, 0, z)$ and $(h_1, 0, d - z)$. The distances are then given by $\overline{AO} \equiv d_1 = \sqrt{h_1^2 + (d - z)^2}$ and $\overline{OB} \equiv d_2 = \sqrt{h_2^2 + z^2}$. The propagation time from A to B is determined by the ray speed v_r as $(d_1 n_{s1} + d_2 n_{s2})/c$, where n_{si} are ray indices for medium i . According to the Fermat's principle, we need to minimize the optical path length defined by $\Delta \equiv d_1 n_{s1} + d_2 n_{s2}$. According to problem (e), we have $n_{s2}^2 = \left(\frac{\overline{OB}}{OB} \cdot \hat{x}_2\right)^2 n_e^2 + \left(\frac{\overline{OB}}{OB} \cdot \hat{z}_2\right)^2 n_o^2$. For an isotropic medium, the ray index is simply the refractive index, i.e., $n_{s1} = n$. Using the following relations

$$\begin{aligned} \frac{\overline{OB}}{OB} \cdot \hat{x}_2 &= \cos(\phi - \theta_2) = \frac{h_2}{d_2} \cos \phi + \frac{z}{d_2} \sin \phi, \\ \frac{\overline{OB}}{OB} \cdot \hat{z}_2 &= \cos\left(\frac{\pi}{2} + \phi - \theta_2\right) = \sin(\theta_2 - \phi) = \frac{z}{d_2} \cos \phi - \frac{h_2}{d_2} \sin \phi, \end{aligned}$$

we find

$$\Delta = n \sqrt{h_1^2 + (d - z)^2} + \sqrt{(h_2 \cos \phi + z \sin \phi)^2 n_e^2 + (-h_2 \sin \phi + z \cos \phi)^2 n_o^2}.$$

The minimum occurs when $\frac{d\Delta}{dz} = 0$. We obtain

$$n \frac{z - d}{\sqrt{h_1^2 + (d - z)^2}} + \frac{(h_2 \sin \phi \cos \phi (n_e^2 - n_o^2) + z(n_e^2 \sin^2 \phi + n_o^2 \cos^2 \phi))}{\sqrt{(h_2 \cos \phi + z \sin \phi)^2 n_e^2 + (-h_2 \sin \phi + z \cos \phi)^2 n_o^2}} = 0.$$

Recognizing $\frac{d-z}{\sqrt{h_1^2+(d-z)^2}} = \sin \theta_1$, moving the second term to the left and taking square of the equation, we obtain

$$n^2 \sin^2 \theta_1 = \frac{(P_3 - P_1 \tan \theta_2)^2}{P_1 \tan^2 \theta_2 - 2P_3 \tan \theta_2 + P_2},$$

where $P_1 = n_o^2 \cos^2 \phi + n_e^2 \sin^2 \phi$, $P_2 = n_o^2 \sin^2 \phi + n_e^2 \cos^2 \phi$, and $P_3 = (n_o^2 - n_e^2) \sin \phi \cos \phi$.

By expanding the above equation out, we find

$$P_1(n^2 \sin^2 \theta_1 - P_1) \tan^2 \theta_2 - 2P_3(n^2 \sin^2 \theta_1 - P_1) \tan \theta_2 + P_2 n^2 \sin^2 \theta_1 - P_3^2 = 0.$$

Hence $\bar{A} = P_1(n^2 \sin^2 \theta_1 - P_1)$, $\bar{B} = -2P_3(n^2 \sin^2 \theta_1 - P_1)$, and $\bar{C} = P_2 n^2 \sin^2 \theta_1 - P_3^2$.

For $\phi = 0$, we have $P_3 = 0$, $P_1 = n_o^2$, and $P_2 = n_e^2$. We find $n_o^2(n^2 \sin^2 \theta_1 - n_o^2) \tan^2 \theta_2 + n_e^2 n^2 \sin^2 \theta_1 = 0$. Hence $\tan \theta_2 = \frac{nm_e \sin \theta_1}{n_o \sqrt{n_o^2 - n^2 \sin^2 \theta_1}}$.

For $\phi = \pi/2$, we have $P_3 = 0$, $P_1 = n_e^2$, and $P_2 = n_o^2$. We find $n_e^2(n^2 \sin^2 \theta_1 - n_e^2) \tan^2 \theta_2 + n_o^2 n^2 \sin^2 \theta_1 = 0$. Hence $\tan \theta_2 = \frac{nm_o \sin \theta_1}{n_e \sqrt{n_e^2 - n^2 \sin^2 \theta_1}}$.

Part C. Entanglement of light

C.1 0.8 pt

Ans:(1) $\omega = \omega_1 \pm \omega_2$, $\vec{k} = \vec{k}_1 \pm \vec{k}_2$

(2) $\hbar\omega = \hbar\omega_1 \pm \hbar\omega_2$, $\hbar\vec{k} = \hbar\vec{k}_1 \pm \hbar\vec{k}_2$ represents the energy conservation and momentum conservation of photons.

(3) Splitting of photon: Energy conservation $\omega = \omega_1 + \omega_2$, momentum conservation: $\vec{k} = \vec{k}_1 + \vec{k}_2$.

Solution:

For a light wave with frequency ω and \vec{k} , the corresponding polarization density and the electric field are in the form of $\vec{A} \cos(\omega t - \vec{k} \cdot \vec{r})$, which can be rewritten as $\frac{\vec{A}}{2}(e^{i(\omega t - \vec{k} \cdot \vec{r})} + e^{-i(\omega t - \vec{k} \cdot \vec{r})})$. By substituting the above form into the equation $P_i^{NL} = \sum_j \sum_k \chi_{ijk}^{(2)} E_j E_k$ and equating the relevant exponents, we find all possible relations are

$$\begin{aligned} \omega &= \omega_1 + \omega_2, \vec{k} = \vec{k}_1 + \vec{k}_2. \\ \text{or } \omega &= \omega_1 - \omega_2, \vec{k} = \vec{k}_1 - \vec{k}_2, \end{aligned}$$

where we have made use of the fact that the frequency is positive. The meaning for the these relations is clear if one recall that the energy and momentum of a photon is given by $\hbar\omega$ and $\hbar\vec{k}$. The relation of $\hbar\omega = \hbar\omega_1 + \hbar\omega_2$, $\hbar\vec{k} = \hbar\vec{k}_1 + \hbar\vec{k}_2$ represents the energy and momentum

conservations when a photon with (ω, \vec{k}) is annihilated and split into two photons with (ω_1, \vec{k}_1) and (ω_2, \vec{k}_2) , while the relation of $\hbar\omega = \hbar\omega_1 + \hbar\omega_2$, $\hbar\vec{k} = \hbar\vec{k}_1 + \hbar\vec{k}_2$ represents the energy and momentum conservations when a photon with (ω_1, \vec{k}_1) is annihilated and split into two photons with (ω, \vec{k}) and (ω_2, \vec{k}_2) .

C.2 0.8 pt

Ans: $\mathbf{o} \rightarrow \mathbf{o} + \mathbf{o}$, $\mathbf{e} \rightarrow \mathbf{e} + \mathbf{e}$

Solution:

For the collinear case, the phase matching conditions become $\omega = \omega_1 + \omega_2$, $\frac{n_i(\omega)\omega}{c} = \frac{n_j(\omega_1)\omega_1}{c} + \frac{n_k(\omega_2)\omega_2}{c}$, where i, j , and k are indices of either \mathbf{o} or \mathbf{e} . Assuming that $\omega_1 \geq \omega_2$, one can solve ω_1 as $\omega_1 = \omega - \omega_2$. We obtain

$$n_i(\omega) - n_j(\omega_1) = \frac{\omega_2}{\omega} [n_k(\omega_2) - n_j(\omega_1)]. \quad (2)$$

Clearly, because $\omega > \omega_1 \geq \omega_2$, if $i = j = k$, $n_i(\omega) - n_j(\omega_1) > 0$ and $n_k(\omega_2) - n_j(\omega_1) \leq 0$, the above equation cannot be satisfied. For other cases, because there is no relation between n_o and n_e , the phase matching conditions can be satisfied. Hence only $\mathbf{o} \rightarrow \mathbf{o} + \mathbf{o}$ and $\mathbf{e} \rightarrow \mathbf{e} + \mathbf{e}$ are not possible.

C.3 1.5 pt

Ans: (1) $M = \frac{K_o[1 - N_e(\Omega_e, \theta) \cot \theta] + K_e}{2K_e K_o}$, $E = -N_e/2M$ and $F = -(\Omega - \Omega_e)(\frac{1}{u_o} - \frac{1}{u_e}) + \frac{N_e^2}{4M}$

(2) the angle between the axis of the cone and z' is $N/K_o = -\frac{2K_e N_e}{K_o[1 - N_e(\Omega_e, \theta) \cot \theta] + K_e}$

(3) the angle of cone is about $\frac{\sqrt{L/M}}{K_o} = -\frac{(\Omega - \Omega_e)}{MK_o}(\frac{1}{u_o} - \frac{1}{u_e}) + \frac{N_e^2}{4M^2 K_o}$.

Solution:

To satisfy the phase matching condition, we expand the angular frequencies ω_1 and ω_2 into $\omega_1 = \Omega_e + \nu$ and $\omega_2 = \Omega_o + \nu'$. Clearly, because $\Omega_e + \Omega_o = \Omega_p$, to satisfy $\omega_1 + \omega_2 = \omega$, $\nu' = -\nu$. Similarly, the conditions for the wavevectors, $\vec{k} = \vec{k}_1 + \vec{k}_2$, can be written as $k_z = k = K_p = k_{1z} + k_{2z}$ and $\vec{k}_{2\perp} = -\vec{k}_{1\perp} \equiv \vec{q}_\perp$. For the \mathbf{o} light ray, we have $k_{2\perp}^2 + k_{2z}^2 = k_2^2$ with $k_2 = \frac{n_o(\omega_2)\omega_2}{c}$. One finds that $k_{2z} = \sqrt{k_2^2 - k_{2\perp}^2} = k_2 - \frac{k_{2\perp}^2}{2k_2}$. Expanding the dependence of ω_2 in k_2 to ν , we obtain

$$k_2 = \frac{n_o(\omega_2)\omega_2}{c} = \frac{n_o(\Omega_o)\Omega_o}{c} + \frac{dk_2}{d\omega_2}(\omega_2 - \Omega_o) = K_o - \frac{\nu}{u_o},$$

where u_o is the group velocity for the ordinary ray. Hence to the second order of corrections,

we get

$$k_{2z} = K_o - \frac{\nu}{u_o} - \frac{q_{\perp}^2}{2K_o}.$$

Similarly, for the **e** light ray, we have $k_{1\perp}^2 + k_{1z}^2 = k_1^2$ with $k_1 = \frac{n_e(\omega_1, \theta_p)\omega_1}{c}$. One finds that $k_{1z} = \sqrt{k_1^2 - k_{1\perp}^2} = k_1 - \frac{k_{1\perp}^2}{2k_1}$. The expansion of k_1 is different from that for k_2 due to its angle dependence. Let the spherical angles for \vec{k}_1 be θ_1 and ϕ_1 . We have

$$k_1 = \frac{n_e(\omega_1, \theta_1)\omega_1}{c} = \frac{n_e(\Omega_e, \theta)\Omega_e}{c} + \frac{dk_1(\Omega_e, \theta)}{d\Omega_e}(\omega_1 - \Omega_e) + \frac{\Omega_e}{c} \frac{dn_e(\Omega_e, \theta)}{d\theta}(\theta_1 - \theta) + \dots$$

Here $\frac{n_e(\Omega_e, \theta)\Omega_e}{c} = K_e$, $\frac{dk_1(\Omega_e, \theta)}{d\Omega_e}$ is $1/u_e$ with u_e being the group velocity for the extraordinary ray and is given by

$$\frac{dk_1(\Omega_e, \theta)}{d\Omega_e} = \frac{n_e(\Omega_e, \theta)}{c} + \frac{\Omega_e}{c} \frac{dn_e(\Omega_e, \theta)}{d\Omega_e}.$$

Because $\frac{dn_e(\Omega_e, \theta)}{d\theta} = \frac{n_o n_e (n_e^2 - n_o^2) \sin \theta \cos \theta}{(n_o^2 \sin^2 \theta + n_e^2 \cos^2 \theta)^{3/2}} = n_e(\Omega_e, \theta) N_e(\Omega_e, \theta)$, we find $N_e(\Omega_e, \theta) = \frac{(n_e^2 - n_o^2) \sin \theta \cos \theta}{n_o^2 \sin^2 \theta + n_e^2 \cos^2 \theta}$. Note that for $n_e < n_o$, $N_e(\Omega_e, \theta) < 0$. To find $\delta\theta = \theta_1 - \theta$, we note that for any \vec{k}_α , one has (cf. Fig. 2(a))

$$\hat{k}_\alpha \cdot \widehat{OA} = \cos \theta_\alpha = \cos \theta \cos \psi_\alpha + \sin \theta \sin \psi_\alpha \cos \phi_\alpha.$$

Since $\sin \psi_1 = |\vec{k}_{\perp,1}|/|\vec{k}_1| = q_{\perp}/k_1 \ll 1$ and $\cos \psi_1 = \sqrt{1 - \sin^2 \psi_1} = 1 - 1/2 \sin^2 \psi_1 + \dots$, to the second order, we can replace k_1 by K_e and obtain

$$\hat{k}_1 \cdot \widehat{OA} = \cos \theta_1 = \cos \theta \left[1 - \frac{1}{2} \frac{q_{\perp}^2}{K_e^2} + \dots \right] + \sin \theta \left[\frac{q_{\perp}}{K_e} + \dots \right] \cos \phi_1.$$

On the other hand, $\cos \theta_1 = \cos \theta + \frac{d \cos \theta}{d\theta}(\theta_1 - \theta) + \dots = \cos \theta - \sin \theta(\theta_1 - \theta) + \dots$. Comparing this equation to the equaton for $\hat{k}_1 \cdot \widehat{OA}$, we obtain

$$\theta_1 - \theta = \frac{1}{2} \frac{q_{\perp}^2}{K_e^2} \cot \theta - \frac{q_{\perp}}{K_e} \cos \phi_1 \dots = \frac{1}{2} \frac{q_{\perp}^2}{K_e^2} \cot \theta + \frac{q_{x'}}{K_e} + \dots$$

Putting all together, we find

$$k_{1z} = K_e + \frac{1}{u_e}(\Omega - \Omega_e) + N_e(\Omega_e, \theta)q_{x'} + \frac{q_{\perp}^2}{2K_e} [N_e(\Omega_e, \theta) \cot \theta - 1] + \dots.$$

The above equation when combined with the equation of k_{1z} and the relation $K_p = k_{1z} + k_{2z}$, we find

$$(\Omega - \Omega_e) \left(\frac{1}{u_e} - \frac{1}{u_o} \right) + N_e(\Omega_e, \theta)q_{x'} + q_{\perp}^2 \left\{ \frac{K_o [N_e(\Omega_e, \theta) \cot \theta - 1] - K_e}{2K_e K_o} \right\} = 0.$$

Because $n_e < n_o$, $N_e(\Omega_e, \theta) < 0$. The above equation can be rewritten in the form

$$M \left[q_{x'} - \frac{N_e}{2D} \right]^2 + M q_{y'}^2 = -(\Omega - \Omega_e) \left(\frac{1}{u_o} - \frac{1}{u_e} \right) + \frac{N_e^2}{4M}.$$

Here $D = \frac{K_o[1 - N_e(\Omega_e, \theta) \cot \theta] + K_e}{2K_e K_o} > 0$. Hence $E = -N_e/2M > 0$ ($N_e < 0$) and $L = -(\Omega - \Omega_e) \left(\frac{1}{u_o} - \frac{1}{u_e} \right) + \frac{N_e^2}{4M}$. Clearly, the cone axis formed by \vec{k}_2 is characterized by \vec{q}_\perp . We find that the angle between the axis of the cone and z' is $\tan^{-1}(N/k_{1z})$, which is about $N/k_{1z} \approx N/K_o = -\frac{2K_e N_e}{K_o[1 - N_e(\Omega_e, \theta) \cot \theta] + K_e}$. The angle of the cone is given by $\sin^{-1} \frac{\sqrt{L/M}}{k_2} \approx \frac{\sqrt{L/M}}{K_o} = -\frac{(\Omega - \Omega_e)}{MK_o} \left(\frac{1}{u_o} - \frac{1}{u_e} \right) + \frac{N_e^2}{4M^2 K_o}$.

C.4 0.8pt

Ans: $P(\alpha, \beta) = \frac{1}{2} \sin^2(\alpha + \beta)$, $P(\alpha, \beta_\perp) = \frac{1}{2} \cos^2(\alpha + \beta)$, $P(\alpha_\perp, \beta) = \frac{1}{2} \cos^2(\alpha + \beta)$, $P(\alpha_\perp, \beta_\perp) = \frac{1}{2} \sin^2(\alpha + \beta)$

Solution:

For a -photon, let the electric field along the polarizer and perpendicular to the polarization represented by $|\alpha_x\rangle$ and $|\alpha_y\rangle$. Here α_x and α_y are essentially the electric field amplitudes in appropriate units. The electric fields (the states) along \hat{x}' and \hat{y}' can be written as

$$\begin{aligned} |\hat{x}'_a\rangle &= \cos \alpha |\alpha_x\rangle - \sin \alpha |\alpha_y\rangle, \\ |\hat{y}'_a\rangle &= \sin \alpha |\alpha_x\rangle + \cos \alpha |\alpha_y\rangle. \end{aligned}$$

Similarly, for b -photon, we have

$$\begin{aligned} |\hat{x}'_b\rangle &= \cos \beta |\beta_x\rangle - \sin \beta |\beta_y\rangle, \\ |\hat{y}'_b\rangle &= \sin \beta |\beta_x\rangle + \cos \beta |\beta_y\rangle. \end{aligned}$$

Hence we obtain

$$\begin{aligned} |\hat{x}'_a\rangle |\hat{y}'_b\rangle &= (\cos \alpha |\alpha_x\rangle - \sin \alpha |\alpha_y\rangle) (\sin \beta |\beta_x\rangle + \cos \beta |\beta_y\rangle), \\ |\hat{y}'_a\rangle |\hat{x}'_b\rangle &= (\sin \alpha |\alpha_x\rangle + \cos \alpha |\alpha_y\rangle) (\cos \beta |\beta_x\rangle - \sin \beta |\beta_y\rangle). \end{aligned}$$

The state of the entangled photon pair can be written as

$$\begin{aligned} & \frac{1}{\sqrt{2}} (|\hat{x}'_a\rangle |\hat{y}'_b\rangle + |\hat{y}'_a\rangle |\hat{x}'_b\rangle) \\ &= \frac{1}{\sqrt{2}} [(\cos \alpha \sin \beta + \sin \alpha \cos \beta) (|\alpha_x\rangle |\beta_x\rangle - |\alpha_y\rangle |\beta_y\rangle) \\ &+ (\cos \alpha \cos \beta - \sin \alpha \sin \beta) (|\alpha_x\rangle |\beta_y\rangle - |\alpha_y\rangle |\beta_x\rangle)] \\ &= \frac{1}{\sqrt{2}} [\sin(\alpha + \beta) (|\alpha_x\rangle |\beta_x\rangle - |\alpha_y\rangle |\beta_y\rangle) + \cos(\alpha + \beta) (|\alpha_x\rangle |\beta_y\rangle - |\alpha_y\rangle |\beta_x\rangle)] \end{aligned}$$

From the above equation, we obtain

$$\begin{aligned}
 P(\alpha, \beta) &= \frac{1}{2} \sin^2(\alpha + \beta), \\
 P(\alpha_{\perp}, \beta_{\perp}) &= \frac{1}{2} \sin^2(\alpha + \beta), \\
 P(\alpha, \beta_{\perp}) &= \frac{1}{2} \cos^2(\alpha + \beta), \\
 P(\alpha_{\perp}, \beta) &= \frac{1}{2} \cos^2(\alpha + \beta).
 \end{aligned}$$

C.5 0.5pt

Ans: $S = |\cos 2(\alpha - \beta) - \cos 2(\alpha - \beta')| + |\cos 2(\alpha' - \beta) + \cos 2(\alpha' - \beta')|$

$S = 2\sqrt{2}$. $S > 2$ indicates that it is not consistent with classical theories.

Solution:

One first realizes that $E(\alpha, \beta) = \frac{P(\alpha, \beta) + P(\alpha_{\perp}, \beta_{\perp}) - P(\alpha, \beta_{\perp}) - P(\alpha_{\perp}, \beta)}{P(\alpha, \beta) + P(\alpha_{\perp}, \beta_{\perp}) + P(\alpha, \beta_{\perp}) + P(\alpha_{\perp}, \beta)}$. Using expressions for P , we find

$$\begin{aligned}
 E(\alpha, \beta) &= \sin^2(\alpha + \beta) - \cos^2(\alpha + \beta) \\
 &= (\sin \alpha \cos \beta + \cos \alpha \sin \beta)^2 - (\cos \alpha \cos \beta - \sin \alpha \sin \beta)^2 \\
 &= -(\cos^2 \alpha - \sin^2 \alpha)(\cos^2 \beta - \sin^2 \beta) + 4 \sin \alpha \sin \beta \cos \alpha \cos \beta \\
 &= \sin(2\alpha) \sin(2\beta) - \cos(2\alpha) \cos(2\beta) = -\cos 2(\alpha - \beta).
 \end{aligned}$$

Hence $S = |\cos 2(\alpha - \beta) - \cos 2(\alpha - \beta')| + |\cos 2(\alpha' - \beta) + \cos 2(\alpha' - \beta')|$. For $\alpha = \frac{\pi}{4}$, $\alpha' = 0$, $\beta = -\frac{\pi}{8}$, $\beta' = \frac{\pi}{8}$, we find $S = |-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}| + |\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}| = 2\sqrt{2} > 2$. Hence classical theories do not apply.