Solution

Water Hammer

Part A. *Excess Pressure and Propagation of Pressure wave*

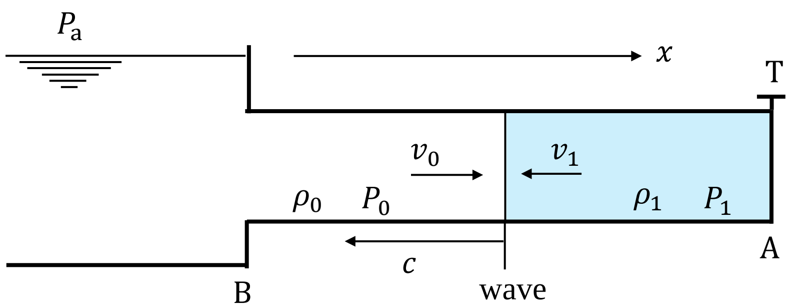


Fig. S1. Pressure wave (shaded) with speed c

A.1 (1.6 pt) Excess pressure and speed of propagation of the pressure wave

When the valve opening is suddenly blocked, fluid pressure at the valve jumps from P_0 to $P_1 = P_0 + \Delta P_s$, thus sending a pressure wave traveling upstream (to the left) with speed c and amplitude ΔP_s. Taking positive x direction as pointing to the right, the velocity of fluid particles next to the valve changes from v_0 to v_1 ($v_1 \leq 0$). Thus the velocity change is $\Delta v = v_1 - v_0$.

In a frame moving to left (along $-x$ direction) with speed c, i.e., riding on the wave (see Fig. S1), velocity of fluid in the pressure wave is $c + v_1$, while that of the incoming fluid in the steady flow ahead of the wave is $c + v_0$. Let ρ_1 be the density of fluid in the pressure wave. From conservation of mass, i.e., equation of continuity, we have

$$\rho_0(c + v_0) = \rho_1(c + v_1) \qquad \text{(a1)}$$

or, by letting $\Delta\rho \equiv \rho_1 - \rho_0$,

$$\frac{\Delta\rho}{\rho_1} = 1 - \frac{\rho_0}{\rho_1} = \frac{v_0 - v_1}{c + v_0} = \frac{-\Delta v}{c + v_0} \qquad \text{(a2)}$$

Moreover, impulse imparted to the fluid must equal its momentum change. Thus, in a short time interval τ after the valve is closed, we must have

$$\rho_0(c + v_0)\tau[(c + v_1) - (c + v_0)] = -\tau\Delta P = (P_0 - P_1)\tau \qquad \text{(a3)}$$

or

$$\Delta P_s = -\rho_0 c\left(1 + \frac{v_0}{c}\right)(v_1 - v_0) = -\rho_0 c\left(1 + \frac{v_0}{c}\right)\Delta v \quad \Rightarrow \quad \alpha = -\left(1 + \frac{v_0}{c}\right) \qquad \text{(a4)}$$

If $v_0/c \ll 1$, we have

$$\Delta P_s = -\rho_0 c \Delta v \qquad \text{(a5)}$$

Note that the *negative* sign in Eqs. (a4) and (a5) follows from the fact that the direction of propagation is opposite to the positive direction for x axis (and velocity). Otherwise the sign should be *positive*. Note also that for a compressional wave

($\Delta P_s > 0$), the velocity imparted to the fluid particle is in the direction of propagation, while for an extensional wave ($\Delta P_s < 0$), the velocity imparted is in the opposite direction of propagation.

Eqs. (a2) and (a4) can be combined to give

$$\Delta P_s = \rho_0 c^2 \left(1 + \frac{v_0}{c}\right)^2 \frac{\Delta\rho}{\rho_1} \quad \text{(a6)}$$

From the definition of the bulk modulus B, which is assumed to be constant, it follows

$$\Delta P_s = B\frac{V_0 - V_1}{V_0} = B\frac{1/\rho_0 - 1/\rho_1}{1/\rho_0} = B\frac{\Delta\rho}{\rho_1} \quad \text{(a7)}$$

From Eqs. (a6) and (a7), we obtain

$$\rho_0 c^2 \left(1 + \frac{v_0}{c}\right)^2 = B \quad \text{(a8)}$$

Thus

$$c = \sqrt{\frac{B}{\rho_0}} - v_0 \qquad \Rightarrow \quad \gamma = 1 \qquad \beta = -v_0 \quad \text{(a9)}$$

However, if in the definition of bulk modulus one uses the fractional change of density $\Delta\rho/\rho_0$ instead of $-\Delta V/V_0$, the result is then $\gamma = 1 + \Delta P_s/B$.* Either result is considered valid.

If $v_0/c \ll 1$, we have

$$c = \sqrt{\frac{B}{\rho_0}} \quad \text{(a10)}$$

*The result (a7) is pointed out by Dr. Jaan Kalda.

A.2 (0.6 pt) Values of c and ΔP_s for water flow

Ans:

From Eqs. (a5) and (a10), we have

$$c = \sqrt{B/\rho_0}$$

$$\Delta P_s = \rho_0 c v_0 = v_0\sqrt{\rho_0 B}$$

Putting in the given values $v_0 = 4.0\ \mathrm{m/s}$, $v_1 = 0$, $\rho_0 = 1.0 \times 10^3\ \mathrm{kg/m^3}$, and $B = 2.2 \times 10^9$ Pa, we have

$$c = \sqrt{B/\rho_0} = 1.5 \times 10^3\ \mathrm{m/s} \quad \text{(b1)}$$

$$\Delta P_s = v_0\sqrt{\rho_0 B} = 5.9\ \mathrm{MPa} \quad \text{(b2)}$$

so that ΔP_s is nearly 59 times the standard pressure.

Note that $v_0/c \sim 10^{-3}$ so that the use of approximate formulas (a5) and (a10) is justified when solving tasks in this problem.

Part B. *A Model for the Flow-Control Valve*

(B.1) (1.0 pt) Excess pressure at valve inlet

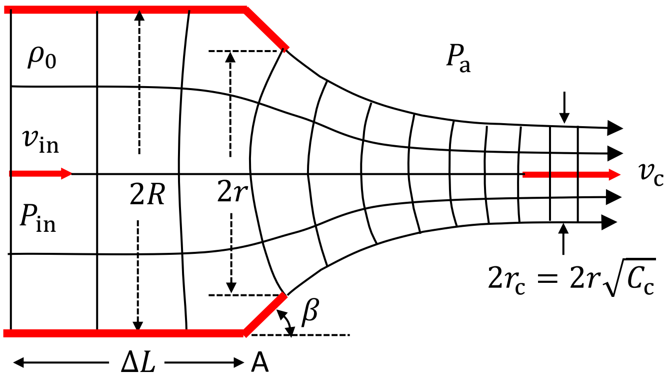


Fig. 2. Valve dimensions and contraction of jet.

Ans:

The model assumes the fluid to be incompressible. Neglecting effects of gravity, Bernoulli's principle gives us

$$\frac{1}{2}\rho_0 v_{in}^2 + P_{in} = \frac{1}{2}\rho_0 v_c^2 + P_a \tag{c1}$$

Equation of continuity and definition of contraction coefficient imply that

$$\pi R^2 v_{in} = \pi r_c^2 v_c = \pi r^2 C_c v_c$$

Therefore

$$v_c = \frac{1}{C_c}\left(\frac{R}{r}\right)^2 v_{in} \tag{c2}$$

From Eqs. (c1) and (c2), we obtain

$$\Delta P_{in} = P_{in} - P_a = \frac{1}{2}\rho_0 v_{in}^2 \left[\frac{1}{C_c^2}\left(\frac{R}{r}\right)^4 - 1\right] = \frac{k}{2}\rho_0 v_{in}^2 \tag{c3}$$

This may be cast into a form involving only dimensionless variables:

$$\frac{\Delta P_{in}}{\rho_0 c^2} = \frac{1}{2}\left(\frac{v_{in}}{c}\right)^2 \left[\frac{1}{C_c^2}\left(\frac{R}{r}\right)^4 - 1\right] = \frac{k}{2}\left(\frac{v_{in}}{c}\right)^2 \tag{c4}$$

where

$$k = \left[\frac{1}{C_c^2}\left(\frac{R}{r}\right)^4 - 1\right] \tag{c5}$$

Thus we see from eq. (c4) that ΔP_{in} is a quadratic function of v_{in}.

Part C. *Water-Hammer Effect due to Fast Closure of Flow-Control Valve*

(C.1) (0.6 pt) Pressure P_0 and velocity v_0 when the valve is fully open

Ans:

According to Bernoulli's theorem and the definition of P_h, we have

$$\frac{1}{2}\rho_0 v_0^2 + P_0 = \frac{1}{2}\rho_0 v_c^2 + P_a = 0 + P_a + \rho_0 gh = P_h \qquad \text{(d1)}$$

From the second equality in the preceding equation, it follows

$$v_c = \sqrt{2gh}$$

Furthermore, from continuity equation and $C_c(r = R) = 1.0$, we have

$$\pi R^2 v_0 = \pi (C_c R)^2 v_c = \pi R^2 v_c \ \Rightarrow\ v_0 = v_c = \sqrt{2gh} \qquad \text{(d2)}$$

Therefore

$$P_0 = P_a = P_h - \rho_0 gh \qquad \text{(d3)}$$

(C.2) (1.2 pt) Pressure $P(t)$ and flow velocity $v(t)$ just before $t = \frac{\tau}{2} = \frac{L}{c}$ and $t = \tau$

Ans:

When the valve is open, the flow in the pipe is steady with velocity v_0 and pressure P_0. The sudden closure of the valve causes an excess pressure ΔP_s on the fluid element next to the valve, causing it to stop with velocity $v_1 = 0$. The velocity change is thus $\Delta v = v_1 - v_0 = -v_0$. Thus, according to Eq. (a5), the excess pressure on the fluid is given by

$$\Delta P_s = -\rho_0 c \Delta v = \rho_0 c v_0 \qquad \text{(e1)}$$

At time $t = \tau/2 = L/c$, the pressure wave reaches the reservoir. The velocity of fluid in the length of the pipe has all changed to $v(\tau/2) = v_1 = v_0 + \Delta v = 0$ and the fluid pressure is $P(\tau/2) = P_1 = P_0 + \Delta P_s = P_0 + \rho_0 c v_0$.

At the reservoir end of the pipe, fluid pressure reduces to the constant hydrostatic pressure $P_h = P_0 + \rho_0 gh$. Equivalently, we may say that the reservoir acts as a free end for the pressure wave and, in reducing its excess pressure to P_h, causes a compression wave to be reflected as an expansion wave. Relative to the hydrostatic pressure P_h, the amplitude of the incoming pressure wave is $\Delta P_{1r} = P_1 - P_h$, hence the reflected expansion wave will have an amplitude $\Delta P_1' = -\Delta P_{1r}$ and we have

$$\Delta P_1' = -\Delta P_{1r} = P_h - P_1 = (P_0 + \rho_0 gh) - (P_0 + \rho_0 c v_0) = -\rho_0 c(v_0 - gh/c) \qquad \text{(e2)}$$

(Here we allow the pressure amplitude to have both signs with negative amplitude signifying an expansion wave.) This will cause the fluid at the reservoir end of the pipe to suffer a velocity change (keeping in mind that the direction of propagation is now the same as the $+x$ axis)

$$\Delta v_{1r} = +\Delta P_1'/(\rho_0 c) = -(v_0 - gh/c)$$

Consequently, its velocity changes to

$$v_{1r} = v_1 + \Delta v_{1r} = 0 - \left(v_0 - \frac{gh}{c}\right) \qquad \text{(e3)}$$

Ahead of the front of the reflected wave, conditions are unchanged and the particle velocity is still $v_1 = 0$ and the fluid pressure is still $P_1 = P_0 + \Delta P_s$, but behind the wave front the particle velocity now becomes $v_{1r} = -(v_0 - gh/c)$ and the pressure becomes

$$P_1 + \Delta P_1' = (P_0 + \rho_0 c v_0) - \rho_0 c \left(v_0 - \frac{gh}{c}\right) = P_0 + \rho_0 gh \qquad \text{(e4)}$$

Therefore, just moment before $t = \tau = 2L/c$ when the front of the reflected wave reaches the valve, the fluid in the whole length of the pipe will be under the pressure $P(\tau) = P_0 + \rho_0 gh = P_h$ as given in Eq. (e4) , and all fluid particles in the pipe will move, as given in Eq. (e3), with velocity $v(\tau) = v_{1\mathrm{r}} = -v_0 + gh/c$, i.e., the fluid in the pipe is expanding and flowing toward the reservoir.

Part D. *Water-Hammer Effect due to Slow Closure of Flow-Control Valve*

(D.1) (3.0 pt) Recursion relations for ΔP_n and v_n

Ans:

Enforcing the approximation $P_h = P_0 + \rho_0 gh \approx P_0$ is equivalent to putting $h = 0$ in all of the results obtained in task (e).

(1) Partial closing $n = 1$

At the valve, immediately after partial closing $n = 1$, fluid pressure jumps from P_0 to P_1, causing flow velocity to change from v_0 to v_1. The pressure and velocity changes are related by Eq. (a5):

$$\frac{1}{\rho_0 c}(P_1 - P_0) = -(v_1 - v_0) \qquad \text{(f1)}$$

Just before reflection by the reservoir, the fluid in the entire pipe has pressure P_1 and velocity v_1. After reflection by the reservoir, i.e., a free end, and before the start of valve closure $n = 2$, the fluid in the entire pipe has pressure (Eq. (e4) with $h = 0$)

$$P_1 - (P_1 - P_0) = P_0$$

and velocity

$$v_1' = v_1 + \frac{-(P_1 - P_0)}{\rho_0 c} = v_1 + (v_1 - v_0)$$

(2) Partial closing $n = 2$

Immediately after partial closing $n = 2$, valve pressure changes from P_0 to P_2, causing flow velocity to change from v_1' to v_2. The pressure and velocity changes are given by Eq. (a5):

$$\frac{1}{\rho_0 c}(P_2 - P_0) = -(v_2 - v_1') = -v_2 + v_1 + (v_1 - v_0) \qquad \text{(f2)}$$

Using Eq. (f1), we may rewrite the preceding equation as

$$\frac{1}{\rho_0 c}(P_2 - P_0) = -(v_2 - v_1) - \frac{1}{\rho_0 c}(P_1 - P_0) \qquad \text{(f3)}$$

Just before reflection by the reservoir, the fluid in the entire pipe has pressure P_2 and velocity v_2. After reflection by the reservoir and before valve closure $n = 3$, the fluid in the entire pipe has pressure

$$P_2 - (P_2 - P_0) = P_0$$

and velocity

$$v_2' = v_2 + (v_2 - v_1')$$

(3) Partial closing $n = 3$

Immediately after partial closing $n = 3$, valve pressure changes from P_0 to P_3, causing flow velocity to change from v_2' to v_3. The pressure and velocity changes are given by Eq. (a5):

$$\frac{1}{\rho_0 c}(P_3 - P_0) = -(v_3 - v_2') = -v_3 + v_2 + (v_2 - v_1') \qquad \text{(f4)}$$

Using Eq. (f2), we may rewrite the preceding equation as

$$\frac{1}{\rho_0 c}(P_3 - P_0) = -(v_3 - v_2) - \frac{1}{\rho_0 c}(P_2 - P_0) \qquad \text{(f5)}$$

Just before reflection by the reservoir, the fluid in the entire pipe has pressure P_3 and velocity v_3. After reflection by the reservoir and before valve closure $n = 4$, the fluid in the entire pipe has pressure

$$P_3 - (P_3 - P_0) = P_0$$

and velocity

$$v_3' = v_3 + (v_3 - v_2')$$

(4) Partial closing $n = 4$

When the valve is fully shut at valve closing $n = 4$, the valve becomes a fixed end, so the fluid velocity at the valve changes from v_3' to $v_4 = 0$. The pressure P_4 at the valve is then given by Eq. (a5):

$$\frac{1}{\rho_0 c}(P_4 - P_0) = -(v_4 - v_3') = -v_4 + v_3 - \frac{1}{\rho_0 c}(P_3 - P_0) \qquad \text{(f6)}$$

Finally, if we take note of the fact that $\Delta P_0 = 0$ and $v_4 = 0$, then all equations obtained above relating excess pressures and velocity changes after valve closings all have the same form:

$$\frac{\Delta P_n}{\rho_0 c} = -(v_n - v_{n-1}) - \frac{\Delta P_{n-1}}{\rho_0 c} \qquad (n = 1,2,3,4) \qquad \text{(f7)}$$

To solve for $\Delta P_n = P_n - P_0$, we note that, from Eqs. (c3) and (c5), we have another relation between ΔP_n and v_n:

$$\Delta P_n = \frac{1}{2} k_n \rho_0 {v_n}^2 \qquad (n = 1,2,3) \qquad \text{(f8)}$$

where C_n represents C_c for $r = r_n$ and

$$k_n = \left[\frac{1}{C_n^2}\left(\frac{R}{r_n}\right)^4 - 1\right] \qquad (n = 1,2,3) \qquad \text{(f9)}$$

Combining Eqs. (f7) and (f8), we have a quadratic equation for v_n:

$$\frac{1}{2} k_n \left(\frac{v_n}{c}\right)^2 + \frac{v_n}{c} + \left(\frac{\Delta P_{n-1}}{\rho_0 c^2} - \frac{v_{n-1}}{c}\right) = 0 \qquad (n = 1,2,3) \qquad \text{(f10)}$$

which can be solved readily using the formula

$$\frac{v_n}{c} = \frac{-1 + \sqrt{1 + 2k_n\left(\frac{v_{n-1}}{c} - \frac{\Delta P_{n-1}}{\rho c^2}\right)}}{k_n} \qquad (n = 1,2,3) \qquad \text{(f11)}$$

If both $\Delta P_{n-1}/(\rho c^2)$ and (v_{n-1}/c) are known, Eq. (f11) may be used to compute v_n/c and then find $\Delta P_n/(\rho c^2)$ by using Eq. (f8). Therefore, Eq. (f7) may be solved iteratively starting with $n = 1$ until $n = 3$. For $n = 4$, we know $v_n = 0$, so Eq. (f7) may be used directly to find ΔP_n.

Note that, from Eq. (f8), ΔP_{n-1} is a quadratic function of v_{n-1}, so that if v_{n-1} is known, then v_n may be computed using Eq. (f11) and then ΔP_n may again be computed using Eq. (f8).

(D.2) (2.0 pt) Estimating ΔP_n and $\rho_0 c v_n$ by graphical method

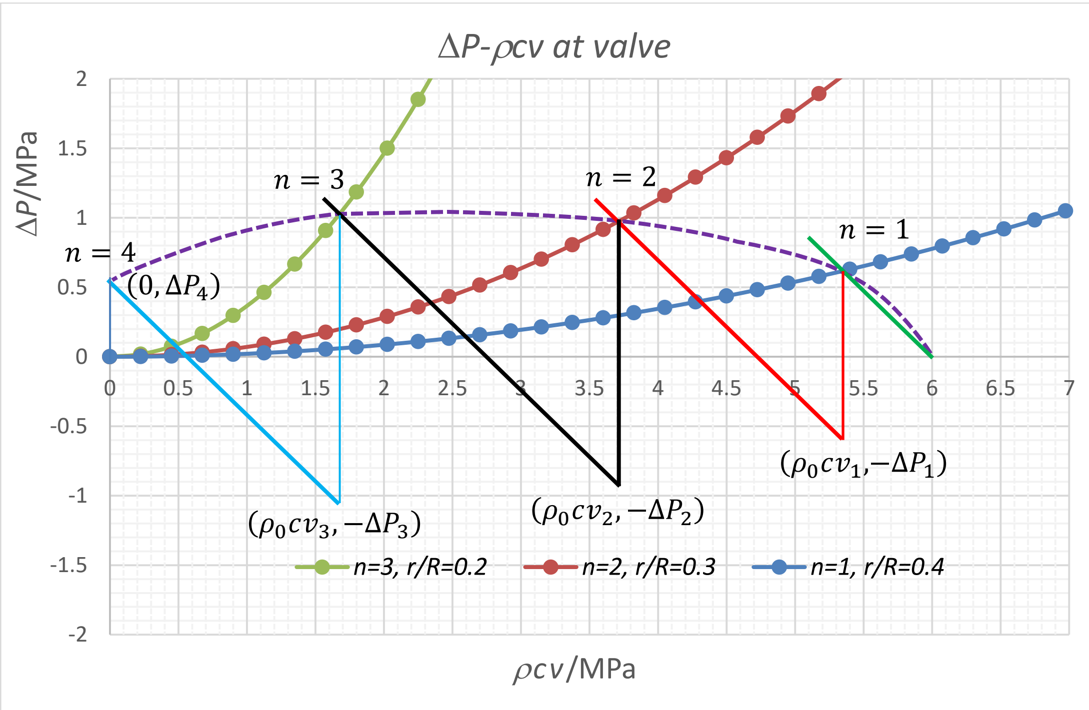
Ans:

To solve Eqs. (f7) and (f8) using graphical method, we rewrite them as follows:

$$\Delta P_n = -(\rho_0 c v_n - \rho_0 c v_{n-1}) - \Delta P_{n-1} \qquad (n = 1,2,3,4) \qquad \text{(g1)}$$

$$\Delta P_n = \frac{k_j}{2\rho_0 c^2}(\rho_0 c v_n)^2 \qquad (n = 1,2,3,4) \qquad \text{(g2)}$$

In a plot of ΔP vs. $\rho_0 c v$, Eq. (g1) and Eq. (g2) correspond to a line passing through the point $(\rho_0 c v_{n-1}, -\Delta P_{n-1})$ with slope -1 and a parabola passing through the origin, respectively. Thus one may readily obtain the solutions for each step of valve closing by locating their points of intersection, starting with $n = 1$. The result is shown in the following graph.



Excess Pressures and particle velocities at the valve for slow closing							
n	r_n/R	C_n	k_n	$v_n/(\text{m/s})$	$\rho_0 c v_n/\text{MPa}$	$\Delta P_n/(\text{MPa})$	$\Delta P_n/(\rho_0 c v_0)$
0	1.00	1.00	0.0	4.0	6.0	0.0	0.0
1	0.40	0.631	97.1	3.6	5.8	0.62	10 %
2	0.30	0.622	318.	2.5	3.8	1.0	17 %
3	0.20	0.616	1646.	1.1	1.7	1.1	18 %

4	0.00			0.0	0.0	0.64	11 %

$\rho_0 c = 1.50 \times 10^6\ \mathrm{kg\ m^{-2}\ s^{-1}}$ $\quad v_0 = 4.0\ \mathrm{m/s}$

Appendix

(The following table and graph are for reference only, not part of the task.)

For $v_0 = 4.0$ m/s, $c = 1.5 \times 10^3$ m/s, and $\rho = 1.0 \times 10^3$ kg/m^3, the results for v_n and ΔP_n are shown in the following table and graph. They are computed according to equations given in task (f). Note that for a sudden full closure of the valve, we have $\Delta P_{\text{sudden}} = \rho c v_0 = 6.0$ MPa.

Excess Pressures and particle velocities at the valve for slow closing							
n	r_n/R	C_n	k_n	$v_n/(\text{m/s})$	$\rho c v_n/\text{MPa}$	$\Delta P_n/(\text{MPa})$	$\Delta P_n/(\rho c v_0)$
0	1.00	1.00	0.0	4.0	6.0	0.0	0.0
1	0.40	0.631	97.1	3.58	5.37	0.624	10 %
2	0.30	0.622	318.	2.50	3.75	0.997	17 %
3	0.20	0.616	1646.	1.13	1.695	1.06	18 %
4	0.00			0.0	0.0	0.643	11 %