17th Asian Physics Olympiad 1-9 May 2016

Theoretical Question – T1

Mechanics of a Deformable Lattice

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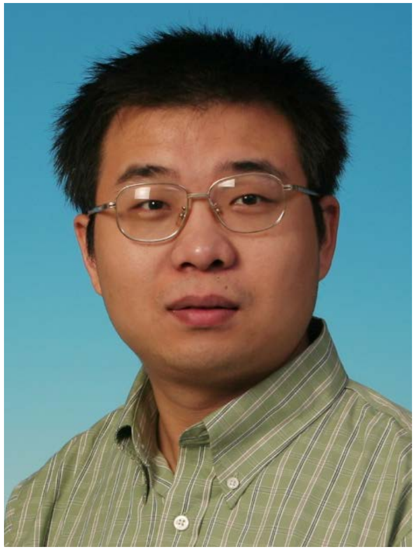
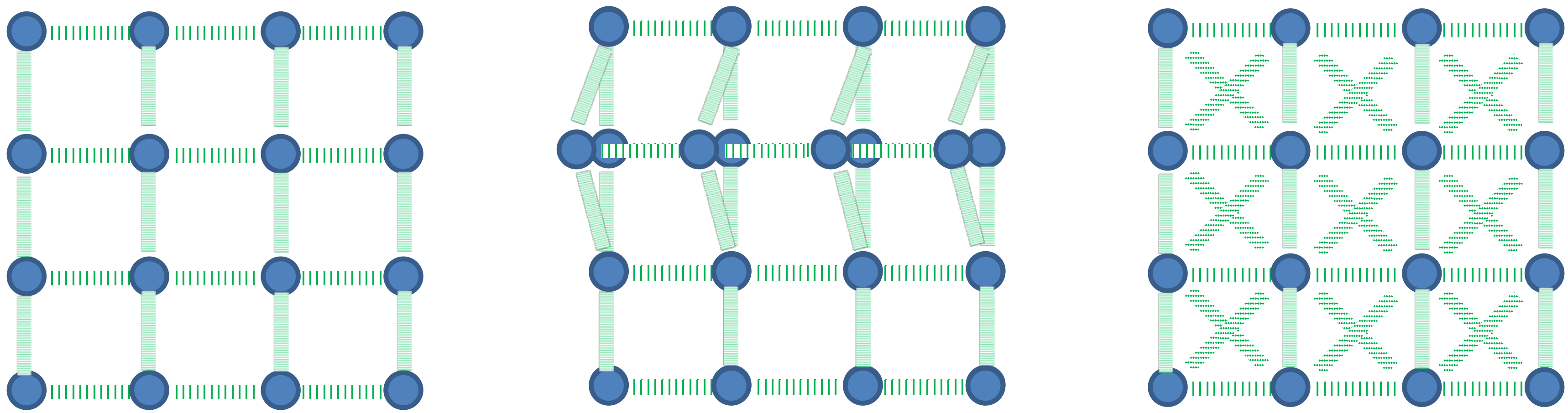


Lattice Stability



Soft floppy mode: zero energy oscillation

James C. Maxwell

Maxwell counting rule

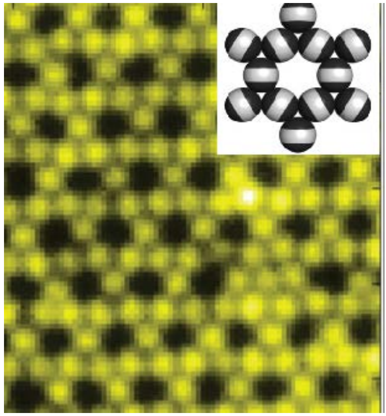
$$N_{fm} \geq dN_s - N_b$$

- dN_s: **Degrees of freedom**
- N_b: **constraints**
- N_{fm}: **# of floppy modes**
- d: **dimension, e.g. d=2**
- N_s: **# of sites**
- N_b: **# of bonds**

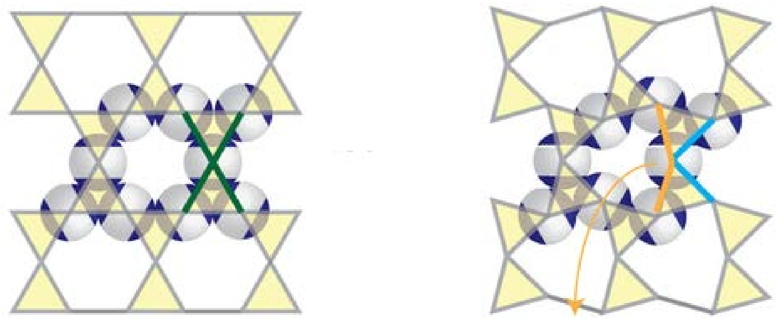
The verge of mechanical stability: $dN_s - N_b$ *= 0*

such ***isostatic lattice*** *has interesting behaviors*

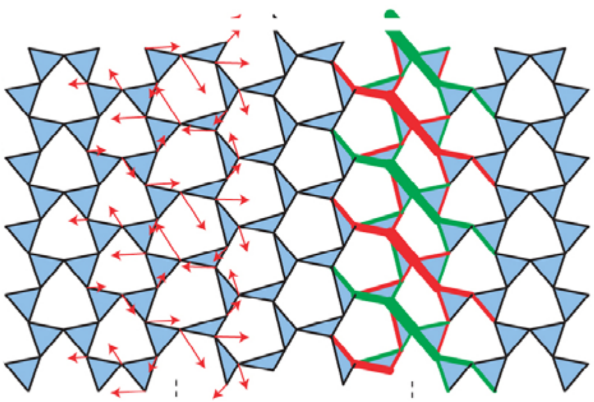
Twists in deformable lattices

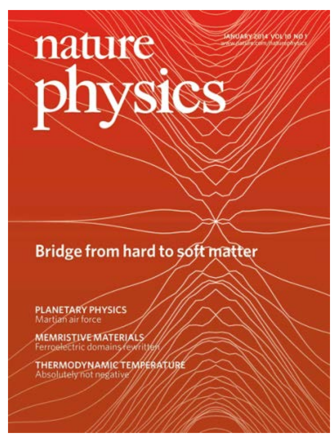


Directed self-assembly of a colloidal kagome lattice, Q. Chen, S. C. Bae & S. Granick, **Nature** 469, 381 (**2011**)

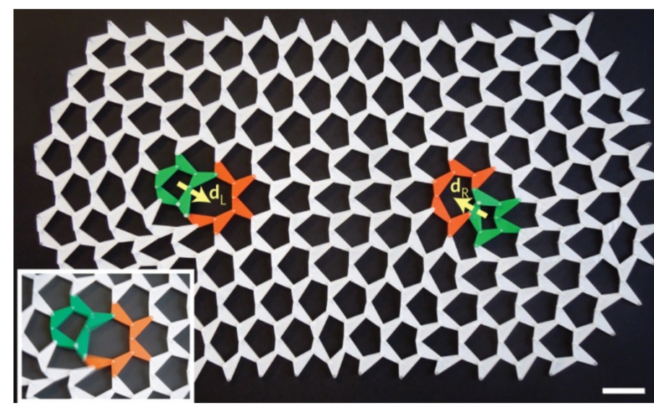


X. Mao, Q. Chen & S. Granick, **Nature Materials** 12, 217 (**2013**)





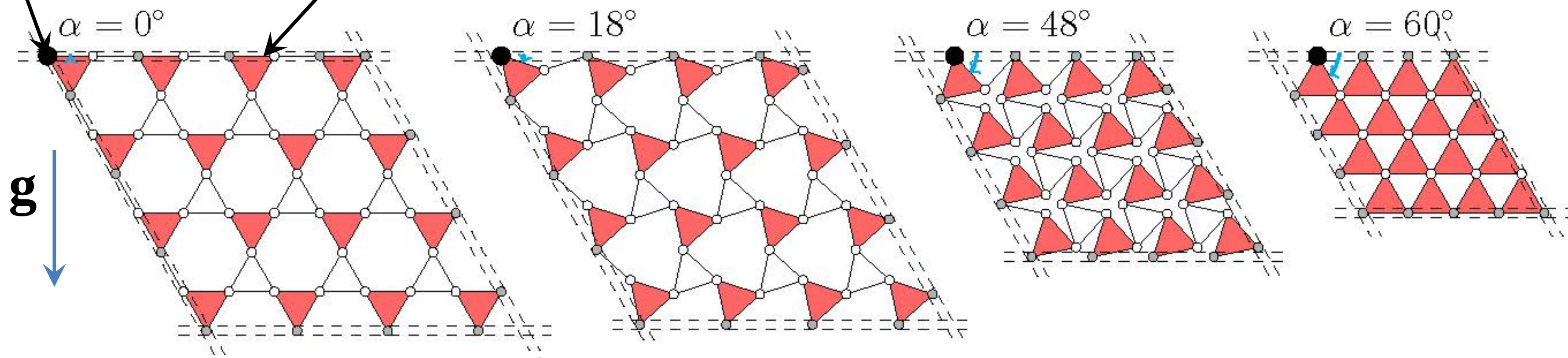
Topological boundary modes in isostatic lattices, **Charles L. Kane & Tom C. Lubensky, Nature Physics** 10, 39 (**2014**).



Topological modes bound to dislocations in mechanical metamaterials, J. Paulose, B. G. Chen & V. Vitelli, **Nature Physics** 11, 153 (**2015**).

A deformable lattice with only one degree of freedom, i.e. can be fully described by one parameter: angle α

fixed point top tube is fixed along the horizontal direction.



1. **N×N red triangles freely hinged by identical rods.**
2. **Each red triangle has mass m. Other parts are massless.**
3. **Four tubes at the four edges are hinged and keep 60° or 120°**
4. **Each tube confines N vertices which can slide in the tube.**
5. **Hanging in gravity like a "curtain" – deformable pendulum**

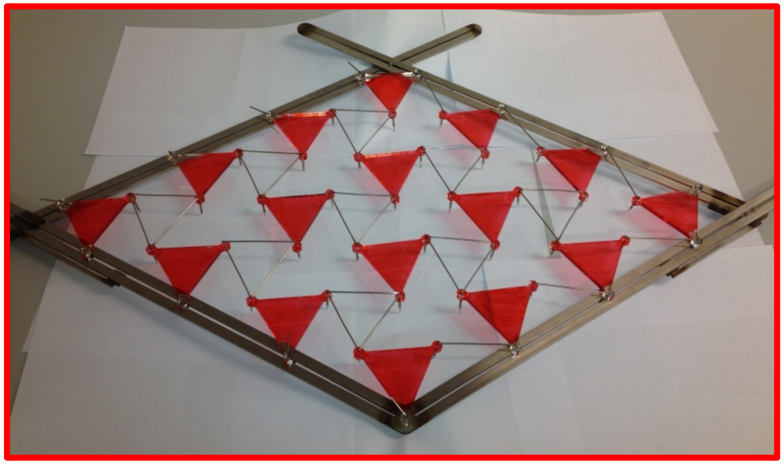
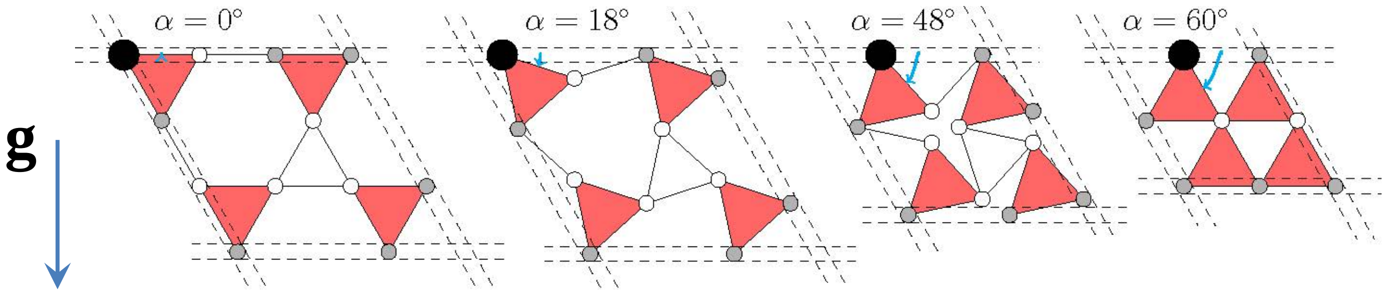


photo of the demo

Part A: 2×2 lattice



Under gravity, what is the equilibrium angle?

Coordinates of vertexes $\Rightarrow$ Potential Energy

The α makes $P.E.(\alpha) = \min \quad \Rightarrow \frac{dP.E.(\alpha)}{d\alpha} = 0 \Rightarrow \alpha$

Applying a small perturbation, what is the frequency?

Total energy $P.E.(\alpha) + K.E.(\alpha)$ follows the form: $\frac{1}{2}K\alpha^2 + \frac{1}{2}I\left(\frac{d\alpha}{dt}\right)^2$

$\Leftrightarrow$ simple harmonic oscillation with angular frequency $\omega = \sqrt{\frac{K}{I}}$

Analogous to a spring: $E = \frac{1}{2}kx^2 + \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 \Rightarrow \omega = \sqrt{\frac{k}{m}}$

Part A: 2×2 lattice

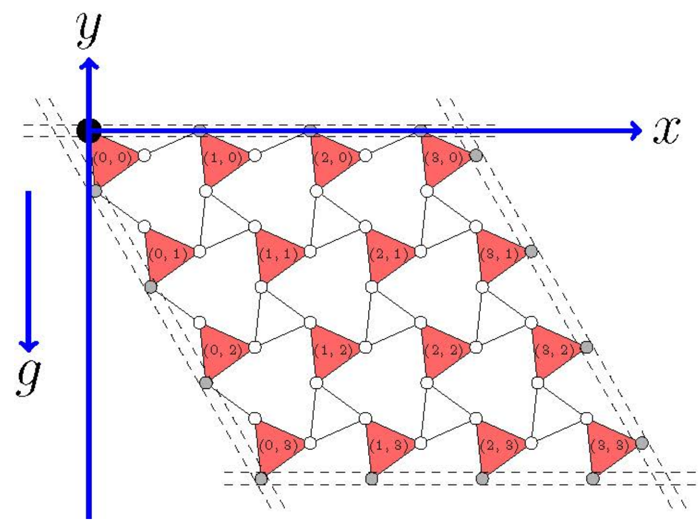
Total **rotational K.E.** relative to each triangle's center of mass:

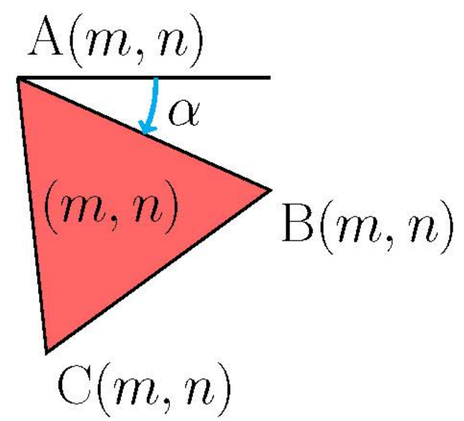
$N^2 \times \frac{1}{2} I(\Delta\dot{\alpha})^2$. $I = \frac{Ml^2}{12}$ is given; its derivation is in Appendix 2.

Total **translational K.E.** of each triangle's center of mass:

coordinates of vertexes of triangle i $\Rightarrow$ center of mass: $x_i(\alpha)$, yi(α)

$$\Rightarrow \sum_i \frac{1}{2} m {v_i}^2 = \sum_i \frac{1}{2} m (\Delta\dot{\alpha})^2 \left[\left(\frac{\mathrm{d}x_i}{\mathrm{d}\alpha}\right)^2 + \left(\frac{\mathrm{d}y_i}{\mathrm{d}\alpha}\right)^2 \right]_{\alpha=\alpha_E}$$





Part B: N×N lattice

Under gravity, what is the equilibrium angle?

Same method as N = 2

$P.E.\sim N^{\gamma_1}$, $K.E.\sim N^{\gamma_2}$, $f\sim N^{\gamma_3}$ **What are γ_1, γ_2, γ_3?**

The y coordinate of the center of mass of N^2 triangles ~ N, so P.E. ~ N^3, i.e. γ_1=3.

K.E. of each triangle ~ 1, so total K.E. ~ N^2, i.e. γ_2 = 2.

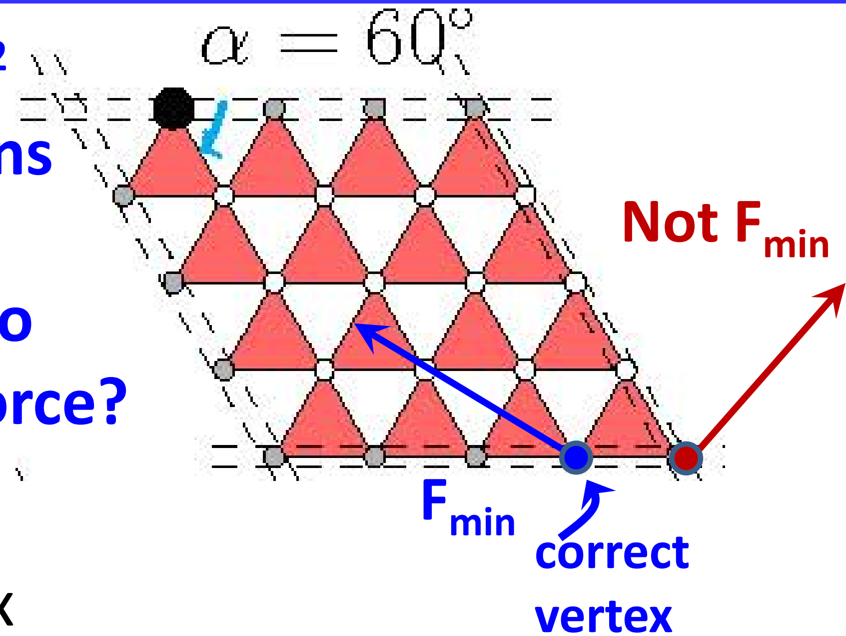
$f_E'\sim\sqrt{\frac{E_p}{E_k}}\sim\sqrt{N}$, **i.e. γ_3 = 0.5.**

Qantitative calculations are in the appendix, similar to N = 2 case.

Part C:

A force is exerted on one of the $3N^2$ vertices so that the system maintains at $\alpha_m = 60°$.

- **Which vertex should we choose to minimize the magnitude of the force?**
- **What is this force?**



The coordinates of an arbitrary vertex

$\Rightarrow$ calculate its displacement when perturbed by $\Delta\alpha$,

$\Rightarrow$ **The vertex with the largest displacement corresponds to the minimum force.**

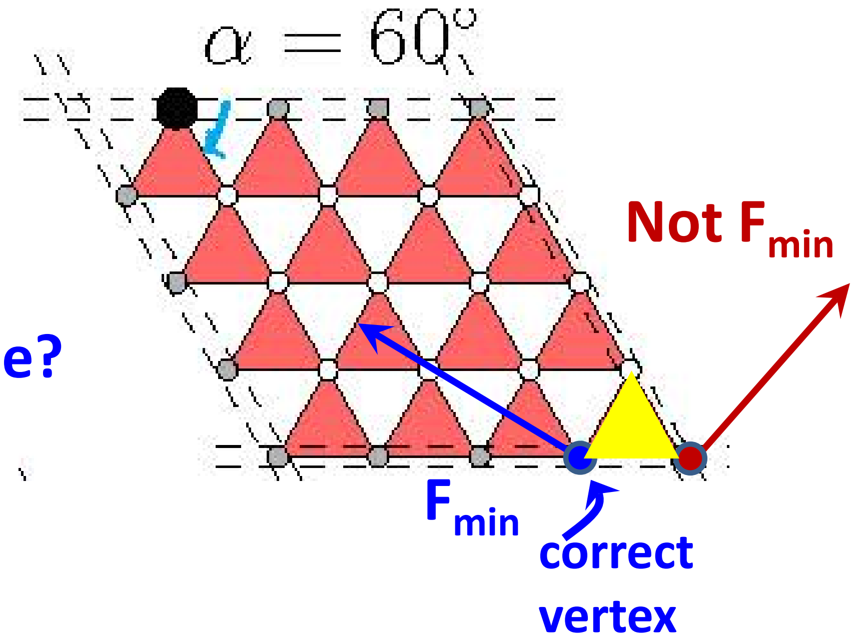
$\Rightarrow$ **Magnitude of the F_{min} = ΔP.E. / max of displacement.**

F_{min} is along the displacement, i.e. in the upper left direction, not an upper right direction (tubes provides constraints, e.g. the top four vertexes are all holding points).

Part C: qualitative argument

A force is exerted on one of the $3N^2$ vertices so that the system maintains at $\alpha_m = 60°$.

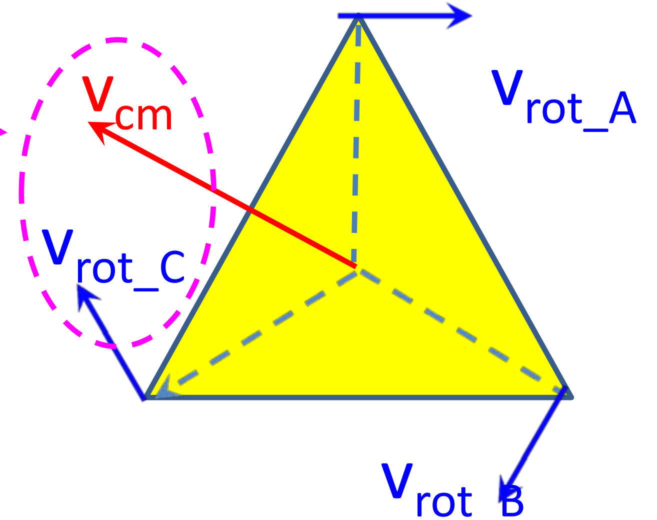
- Which vertex should we choose to minimize the magnitude of the force?
- What is this force?



The farthest triangle:

$$\vec{v} = \vec{v}_{\mathrm{cm}} + \vec{v}_{\mathrm{rot}}$$

Their directions are most close ⇒ Vertex C has the largest displacement, i.e. **the minimum force**



Covered Topics in the Syllabus

- Mechanics
  - Conservation of Energy, Force and Motion, Gravitational Potential Energy
- Mechanics of Rigid Bodies:
  - Center of Mass, Moment of Inertia, Rotational Kinetic Energy, Conditions of Equilibrium
- Oscillation and Waves:
  - Harmonic Oscillations, Perturbations, Angular Frequency