Theoretical 2: Solution
Relativistic Correction on GPS Satelitte

Part A. Single accelerated particle

1. The equation of motion is given by

$$\begin{aligned} F &= \frac{d}{dt}(\gamma m v) \qquad (1) \\ &= \frac{mc\dot{\beta}}{(1-\beta^2)^{\frac{3}{2}}} \\ F &= \gamma^3 m a, \qquad (2) \end{aligned}$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = \frac{v}{c}$. So the acceleration is given by

$$a = \frac{F}{\gamma^3 m}. \qquad (3)$$

2. Eq.(3) can be rewritten as

$$\begin{aligned} c\frac{d\beta}{dt} &= \frac{F}{\gamma^3 m} \\ \int_0^\beta \frac{d\beta}{(1-\beta^2)^{\frac{3}{2}}} &= \frac{F}{mc}\int_0^t dt \\ \frac{\beta}{\sqrt{1-\beta^2}} &= \frac{Ft}{mc} \qquad (4) \\ \beta &= \frac{\frac{Ft}{mc}}{\sqrt{1+\left(\frac{Ft}{mc}\right)^2}}. \qquad (5) \end{aligned}$$

3. Using Eq.(5), we get

$$\begin{aligned} \int_0^x dx &= \int_0^t \frac{Ftdt}{m\sqrt{1+\left(\frac{Ft}{mc}\right)^2}} \\ x &= \frac{mc^2}{F}\left(\sqrt{1+\left(\frac{Ft}{mc}\right)^2} - 1\right). \qquad (6) \end{aligned}$$

4. Consider the following systems, a frame S' is moving with respect to another frame S, with velocity u in the x direction. If a particle is moving in the S' frame with velocity v' also in x direction, then the particle velocity in the S frame is given by

$$v = \frac{u+v'}{1+\frac{uv'}{c^2}}. \qquad (7)$$

If the particles velocity changes with respect to the S' frame, then the velocity in the S frame is also change according to

$$dv = \frac{dv'}{1+\frac{uv'}{c^2}} - \frac{u+v'}{\left(1+\frac{uv'}{c^2}\right)^2}\frac{udv'}{c^2}$$

$$dv = \frac{1}{\gamma^2}\frac{dv'}{\left(1+\frac{uv'}{c^2}\right)^2}. \tag{8}$$

The time in the S' frame is t', so the time in the S frame is given by

$$t = \gamma\left(t' + \frac{ux'}{c^2}\right), \tag{9}$$

so the time change in the S' frame will give a time change in the S frame as follow

$$dt = \gamma dt'\left(1 + \frac{uv'}{c^2}\right). \tag{10}$$

The acceleration in the S frame is given by

$$a = \frac{dv}{dt} = \frac{a'}{\gamma^3}\frac{1}{\left(1+\frac{uv'}{c^2}\right)^3}. \tag{11}$$

If the S' frame is the proper frame, then by definition the velocity $v' = 0$. Substitute this to the last equation, we get

$$a = \frac{a'}{\gamma^3}. \tag{12}$$

Combining Eq.(3) and Eq.(12), we get

$$a' = \frac{F}{m} \equiv g. \tag{13}$$

5. Eq.(3) can also be rewritten as

$$c\frac{d\beta}{\gamma d\tau} = \frac{g}{\gamma^3} \tag{14}$$

$$\int_0^\beta \frac{d\beta}{1-\beta^2} = \frac{g}{c}\int_0^\tau d\tau$$

$$\ln\left(\frac{1}{\sqrt{1-\beta^2}} + \frac{\beta}{\sqrt{1-\beta^2}}\right) = \frac{g\tau}{c} \tag{15}$$

$$\sqrt{\frac{1+\beta}{1-\beta}} = e^{\frac{g\tau}{c}}$$

$$\beta\left(e^{\frac{g\tau}{c}} + e^{-\frac{g\tau}{c}}\right) = e^{\frac{g\tau}{c}} - e^{-\frac{g\tau}{c}}$$

$$\beta = \tanh\frac{g\tau}{c}. \tag{16}$$

6. The time dilation relation is

$$dt = \gamma d\tau. \tag{17}$$

From eq.(16), we have

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh\frac{g\tau}{c}. \tag{18}$$

Combining this equations, we get

$$\int_0^t dt = \int_0^\tau d\tau \cosh\frac{g\tau}{c}$$
$$t = \frac{c}{g}\sinh\frac{g\tau}{c}. \tag{19}$$

Part B. Flight Time

1. When the clock in the origin time is equal to t_0, it emits a signal that contain the information of its time. This signal will arrive at the particle at time t, while the particle position is at $x(t)$. We have

$$c(t-t_0) = x(t) \tag{20}$$
$$t - t_0 = \frac{c}{g}\left(\sqrt{1+\left(\frac{gt}{c}\right)^2}-1\right)$$
$$t = \frac{t_0}{2}\frac{2-\frac{gt_0}{c}}{1-\frac{gt_0}{c}}. \tag{21}$$

When the information arrive at the particle, the particle's clock has a reading according to eq.(19). So we get

$$\frac{c}{g}\sinh\frac{g\tau}{c} = \frac{t_0}{2}\frac{2-\frac{gt_0}{c}}{1-\frac{gt_0}{c}}$$
$$0 = \frac{1}{2}\left(\frac{gt_0}{c}\right)^2 - \frac{gt_0}{c}\left(1+\sinh\frac{g\tau}{c}\right) + \sinh\frac{g\tau}{c}$$
$$\frac{gt_0}{c} = 1 + \sinh\frac{g\tau}{c} \pm \cosh\frac{g\tau}{c}. \tag{22}$$

Using initial condition $t = 0$ when $\tau = 0$, we choose the negative sign

$$\frac{gt_0}{c} = 1 + \sinh\frac{g\tau}{c} - \cosh\frac{g\tau}{c}$$
$$t_0 = \frac{c}{g}\left(1 - e^{-\frac{g\tau}{c}}\right). \tag{23}$$

As $\tau \to \infty$, $t_0 = \frac{c}{g}$. So the clock reading will freeze at this value.

2. When the particles clock has a reading τ_0, its position is given by eq.(6), and the time t_0 is given by eq.(19). Combining this two equation, we get

$$x = \frac{c^2}{g}\left(\sqrt{1+\sinh^2\frac{g\tau_0}{c}}-1\right). \tag{24}$$

The particle's clock reading is then sent to the observer at the origin. The total time needed for the information to arrive is given by

$$t = \frac{c}{g}\sinh\frac{g\tau_0}{c}+\frac{x}{c} \tag{25}$$
$$= \frac{c}{g}\left(\sinh\frac{g\tau_0}{c}+\cosh\frac{g\tau_0}{c}-1\right)$$
$$t = \frac{c}{g}\left(e^{\frac{g\tau_0}{c}}-1\right) \tag{26}$$
$$\tau_0 = \frac{c}{g}\ln\left(\frac{gt}{c}+1\right). \tag{27}$$

The time will not freeze.

Part C. Minkowski Diagram

1. The figure below show the setting of the problem.

The line AB represents the stick with proper length equal L in the S frame.
The length AB is equal to $\sqrt{\frac{1-\beta^2}{1+\beta^2}}L$ in the S' frame.
The stick length in the S' frame is represented by the line AC

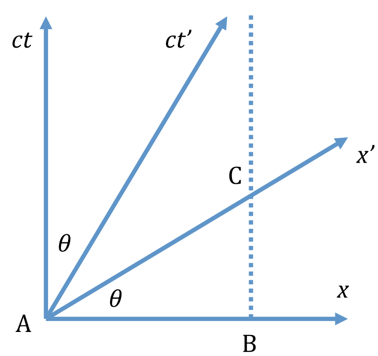

Figure 1: Minkowski Diagram

$$\text{AC} = \frac{\text{AB}}{\cos\theta} = \sqrt{1-\beta^2}L. \tag{28}$$

2. The position of the particle is given by eq.(6).

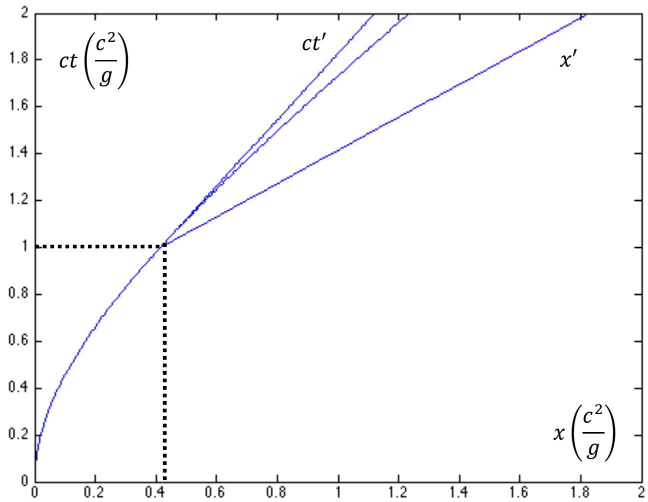

Figure 2: Minkowski Diagram

Part D. Two Accelerated Particles

1. $\tau_B = \tau_A$.
2. From the diagram, we have

$$\tan\theta = \beta = \frac{ct_2 - ct_1}{x_2 - x_1}. \tag{29}$$

Using eq.(6), and eq.(19) along with the initial condition, we get

$$x_1 = \frac{c^2}{g}\left(\cosh\frac{g\tau_1}{c} - 1\right), \tag{30}$$
$$x_2 = \frac{c^2}{g}\left(\cosh\frac{g\tau_2}{c} - 1\right) + L. \tag{31}$$

Using eq.(16), eq.(19), eq.(30) and eq.(31), we obtain

$$\begin{aligned}
\tanh\frac{g\tau_1}{c} &= \frac{c\left(\frac{c}{g}\sinh\frac{g\tau_2}{c} - \frac{c}{g}\sinh\frac{g\tau_1}{c}\right)}{L + \frac{c^2}{g}\left(\cosh\frac{g\tau_2}{c} - 1\right) - \frac{c^2}{g}\left(\cosh\frac{g\tau_1}{c} - 1\right)} \\
&= \frac{\sinh\frac{g\tau_2}{c} - \sinh\frac{g\tau_1}{c}}{\frac{gL}{c^2} + \cosh\frac{g\tau_2}{c} - \cosh\frac{g\tau_1}{c}} \\
\frac{gL}{c^2}\sinh\frac{g\tau_1}{c} &= \sinh\frac{g\tau_2}{c}\cosh\frac{g\tau_1}{c} - \cosh\frac{g\tau_2}{c}\sinh\frac{g\tau_1}{c} \\
\frac{gL}{c^2}\sinh\frac{g\tau_1}{c} &= \sinh\frac{g}{c}\left(\tau_2 - \tau_1\right).
\end{aligned} \tag{32}$$

So $C_1 = \frac{gL}{c^2}$.

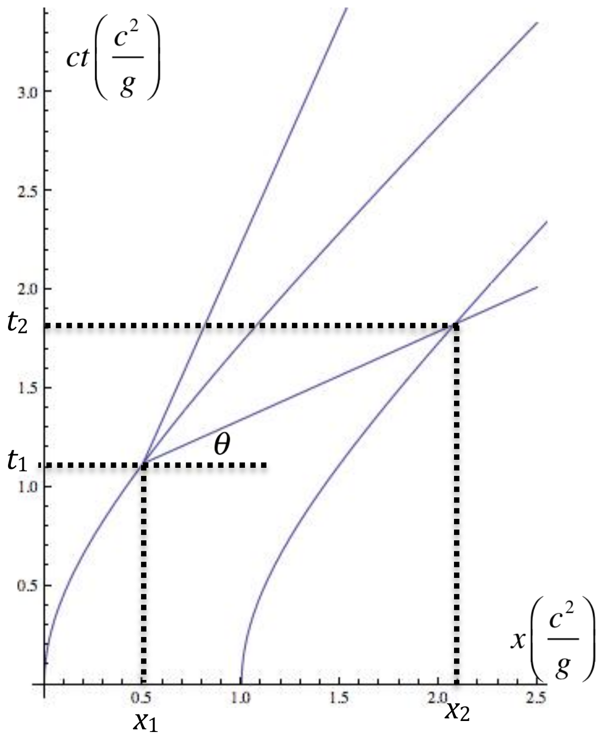

Figure 3: Minkowski Diagram for two particles

3. From the length contraction, we have

$$L' = \frac{x_2 - x_1}{\gamma_1} \tag{33}$$

$$\frac{dL'}{d\tau_1} = \left(\frac{dx_2}{d\tau_2}\frac{d\tau_2}{d\tau_1} - \frac{dx_1}{d\tau_1}\right)\frac{1}{\gamma_1} - \frac{x_2 - x_1}{\gamma_1^2}\frac{d\gamma_1}{d\tau_1}. \tag{34}$$

Take derivative of eq.(30), eq.(31) and eq.(32), we get

$$\frac{dx_1}{d\tau_1} = c\sinh\frac{g\tau_1}{c}, \tag{35}$$

$$\frac{dx_2}{d\tau_2} = c\sinh\frac{g\tau_2}{c}, \tag{36}$$

$$\frac{gL}{c^2}\cosh\frac{g\tau_1}{c} = \cosh\frac{g}{c}\left(\tau_2 - \tau_1\right)\left(\frac{d\tau_2}{d\tau_1} - 1\right). \tag{37}$$

The last equation can be rearrange to get

$$\frac{d\tau_2}{d\tau_1} = \frac{\frac{gL}{c^2}\cosh\frac{g\tau_1}{c}}{\cosh\frac{g}{c}\left(\tau_2 - \tau_1\right)} + 1. \tag{38}$$

From eq.(29), we have

$$x_2 - x_1 = \frac{c(t_2 - t_1)}{\beta_1} = \frac{c}{\tanh \frac{g\tau_1}{c}} \left(\frac{c}{g} \sinh \frac{g\tau_2}{c} - \frac{c}{g} \sinh \frac{g\tau_1}{c} \right). \tag{39}$$

Combining all these equations, we get

$$\begin{aligned} \frac{dL_1}{d\tau_1} &= \left(c \sinh \frac{g\tau_2}{c} \frac{\frac{gL}{c^2} \cosh \frac{g\tau_1}{c}}{\cosh \frac{g}{c}(\tau_2 - \tau_1)} + c \sinh \frac{g\tau_2}{c} - c \sinh \frac{g\tau_1}{c} \right) \frac{1}{\cosh \frac{g\tau_1}{c}} \\ &\quad - \frac{c^2}{g} \left(\sinh \frac{g\tau_2}{c} - \sinh \frac{g\tau_1}{c} \right) \frac{1}{\tanh \frac{g\tau_1}{c}} \frac{1}{\cosh^2 \frac{g\tau_1}{c}} \frac{g}{c} \sinh \frac{g\tau_1}{c} \\ \frac{dL_1}{d\tau_1} &= \frac{gL}{c} \frac{\sinh \frac{g\tau_2}{c}}{\cosh \frac{g}{c}(\tau_2 - \tau_1)}. \end{aligned} \tag{40}$$

So $C_2 = \frac{gL}{c}$.

Part E. Uniformly Accelerated Frame

1. Distance from a certain point x_p according to the particle's frame is

$$L' = \frac{x - x_p}{\gamma} \tag{41}$$

$$L' = \frac{\frac{c^2}{g_1}\left(\cosh \frac{g_1\tau}{c} - 1\right) - x_p}{\cosh \frac{g_1\tau}{c}}$$

$$L' = \frac{c^2}{g_1} - \frac{\frac{c^2}{g_1} + x_p}{\cosh \frac{g_1\tau}{c}}. \tag{42}$$

For L' equal constant, we need $x_p = -\frac{c^2}{g_1}$.

2. **First method:** If the distance in the S' frame is constant $= L$, then in the S frame the length is

$$L_s = L\sqrt{\frac{1+\beta^2}{1-\beta^2}}. \tag{43}$$

So the position of the second particle is

$$\begin{aligned} x_2 &= x_1 + L_s \cos\theta \qquad (44) \\ &= \frac{c^2}{g_1} \left(\sqrt{1 + \left(\frac{g_1 t_1}{c}\right)^2} - 1 \right) + L\sqrt{1 + \left(\frac{g_1 t_1}{c}\right)} \\ x_2 &= \left(\frac{c^2}{g_1} + L \right) \sqrt{1 + \left(\frac{g_1 t_1}{c}\right)^2} - \frac{c^2}{g_1}. \qquad (45) \end{aligned}$$

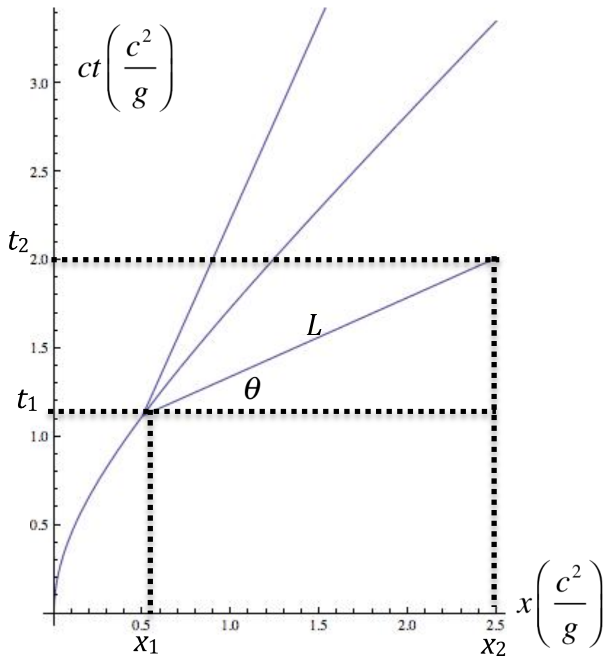

Figure 4: Minkowski Diagram for two particles

The time of the second particle is

$$\begin{aligned} ct_2 &= ct_1 + L_s \sin\theta \\ &= ct_1 + L\sqrt{\frac{1+\beta^2}{1-\beta^2}}\frac{\beta}{\sqrt{1+\beta^2}} \end{aligned} \tag{46}$$

$$ct_2 = t_1\left(c + \frac{g_1 L}{c}\right). \tag{47}$$

Substitute eq.(47) to eq.(45) to get

$$x_2 = \left(\frac{c^2}{g_1} + L\right)\sqrt{1 + \left(\frac{g_1}{c}\frac{t_2}{1+\frac{g_1 L}{c^2}}\right)^2} - \frac{c^2}{g_1}$$

$$x_2 = \left(\frac{c^2}{g_1} + L\right)\sqrt{1 + \left(\frac{g_1}{1+\frac{g_1 L}{c^2}}\frac{t_2}{c}\right)^2} - \frac{c^2}{g_1}. \tag{48}$$

From the last equation, we can identify

$$g_2 \equiv \frac{g_1}{1+\frac{g_1 L}{c^2}}. \tag{49}$$

As for confirmation, we can subsitute this relation to the second particle position to get

$$x_2 = \frac{c^2}{g_2}\sqrt{1+\left(\frac{g_2 t_2}{c}\right)^2} - \frac{c^2}{g_1}. \tag{50}$$

Second method: In this method, we will choose g_2 such that the special point like the one descirbe in the question 1 is exactly the same as the similar point for the proper acceleration g_1.

For first particle, we have $x_{p1}g_1 = c^2$

For second particle, we have $(L + x_{p1})g_2 = c^2$

Combining this two equations, we get

$$\begin{aligned} g_2 &= \frac{c^2}{L+\frac{c^2}{g_1}} \\ g_2 &= \frac{g_1}{1+\frac{g_1 L}{c^2}}. \end{aligned} \tag{51}$$

3. The relation between the time in the two particles is given by eq.(47)

$$\begin{aligned} t_2 &= t_1\left(1+\frac{g_1 L}{c^2}\right) \\ \frac{c^2}{g_2}\sinh\frac{g_2\tau_2}{c} &= \frac{c^2}{g_1}\sinh\frac{g_1\tau_1}{c}\left(1+\frac{g_1 L}{c^2}\right) \\ \sinh\frac{g_2\tau_2}{c} &= \sinh\frac{g_1\tau_1}{c} \end{aligned}$$

$$g_2\tau_2 = g_1\tau_1 \tag{52}$$

$$\frac{d\tau_2}{d\tau_1} = \frac{g_1}{g_2} = 1+\frac{g_1 L}{c^2}. \tag{53}$$

Part F. Correction for GPS

1. From Newtons Law

$$\frac{GMm}{r^2} = m\omega^2 r \tag{54}$$

$$r = \left(\frac{gR^2T^2}{4\pi^2}\right)^{\frac{1}{3}} \tag{55}$$

$$r = 2.66\times 10^7\text{m}.$$

The velocity is given by

$$v = \omega r = \left(\frac{2\pi g R^2}{T}\right)^{\frac{1}{3}} \tag{56}$$

$$= 3.87\times 10^3\text{m/s}.$$

2. The general relativity effect is

$$\frac{d\tau_g}{dt} = 1 + \frac{\Delta U}{mc^2} \tag{57}$$
$$\frac{d\tau_g}{dt} = 1 + \frac{gR^2}{c^2}\frac{R-r}{Rr}. \tag{58}$$

   After one day, the difference is

$$\begin{aligned}\Delta\tau_g &= \frac{gR^2}{c^2}\frac{R-r}{Rr}\Delta T \\ &= 4.55 \times 10^{-5}\text{s}.\end{aligned} \tag{59}$$

   The special relativity effect is

$$\begin{aligned}\frac{d\tau_s}{dt} &= \sqrt{1-\frac{v^2}{c^2}} \\ &= \sqrt{1-\left(\left(\frac{2\pi gR^2}{T}\right)^{\frac{2}{3}}\right)\frac{1}{c^2}}\end{aligned} \tag{60}$$
$$\approx 1 - \frac{1}{2}\left(\left(\frac{2\pi gR^2}{T}\right)^{\frac{2}{3}}\right)\frac{1}{c^2}. \tag{61}$$

   After one day, the difference is

$$\begin{aligned}\Delta\tau_s &= -\frac{1}{2}\left(\left(\frac{2\pi gR^2}{T}\right)^{\frac{2}{3}}\right)\frac{1}{c^2}\Delta T \\ &= -7.18 \times 10^{-6}\text{s}.\end{aligned} \tag{62}$$

   The satelite's clock is faster with total $\Delta\tau = \Delta\tau_g + \Delta\tau_s = 3.83 \times 10^{-5}$s.

3. $\Delta L = c\Delta\tau = 1.15 \times 10^4\text{m} = 11.5\text{km}$.