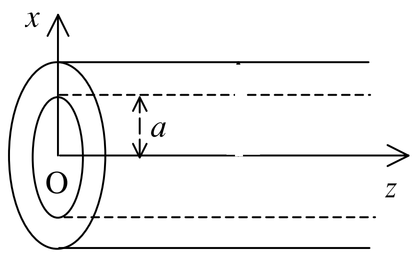
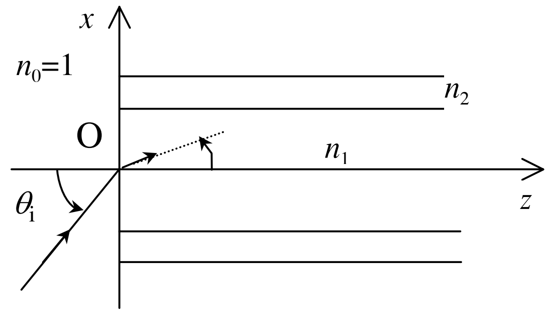
Theoretical Competition-Problem No. 2

Optical fiber





An optical fiber consists of a cylindrical core of radius a, made of a transparent material with refraction index varying gradually from the value $n=n_1$ on the axis to $n = n_2$ (with $1<n_2<n_1$) at a distance a from the axis, according to the formula

$$n = n(x) = n_1\sqrt{1-\alpha^2x^2}$$

where x is the distance from the core axis and α is a constant. The core is surrounded by a cladding made of a material with constant refraction index n_2. Outside the fiber is air, of refractive index n_0.

Let Oz be the axis of the fiber, with O - the center of the fiber end.

Given n_0=1.000; n_1=1.500; n_2= 1.460, a = 25 μm.

1. A monochromatic light ray enters the fiber at point O under an incident angle θ_i, the incident plane being the plane xOz.

a. Show that at each point on the trajectory of the light in the fiber, the refractive index n and the angle θ between the light ray and the Oz axis satisfy the relationship $n\,\boldsymbol{cos}\,\theta = C$ where C is a constant. Find the expression for C in terms of n_1 and θ_i. ***[1.0 points]***

b. Use the result found in 1.a. and the trigonometric relation $\boldsymbol{cos}\,\theta = \left(1+\boldsymbol{tan}^2\,\theta\right)^{-\frac{1}{2}}$, where $\boldsymbol{tan}\,\theta = \dfrac{dx}{dz} = x'$ is the slope of the tangent to the trajectory at point (x, z), derive an equation for x'. Find the full expression for α in terms of n_1, n_2 and a. By differentiating the two sides of this equation versus z, find the equation for the second derivative x''. ***[1.0 points]***

c. Find the expression of x as a function of z , that is $x = f(z)$, which satisfies the above equation. This is the equation of the trajectory of light in the fiber. ***[1.0 points]***

d. Sketch one full period of the trajectories of the light rays entering the fiber under two different incident angles θ_i. ***[1.0 points]***

2. Light propagates in the optical fiber.

a. Find the maximum incident angle θ_{iM} , under which the light ray still can propagate inside the core of the fiber. ***[1.5 points]***

b. Determine the expression for coordinate z of the crossing points of a light ray with Oz axis for $\theta_i \neq 0$. ***[1.5 points]***

3. The light is used to transmit signals in the form of very short light pulses (of negligible pulse width).

a. Determine the time τ it takes the light to travel from point O to the first crossing point with Oz for incident angle $\theta_i \neq 0$ and $\theta_i \leq \theta_{iM}$.

The ratio of the coordinate z of the first crossing point and τ is called the propagation speed of the light signal along the fiber. Assume that this speed varies monotonously with θ_i.

Find this speed (called v_M) for $\theta_i = \theta_{iM}$.

Find also the propagation speed (called v_0) of the light along the axis Oz.

Compare the two speeds. ***[3.25 points]***

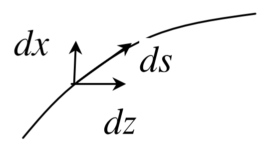
b. The light beam bearing the signals is a converging beam entering the fiber at O under different incident angles θ_i with $0 \leq \theta_i \leq \theta_{iM}$. Calculate the highest repetition frequency f of the signal pulses, so that at a distance z = 1000 m two consecutive pulses are still separated (that is, the pulses do not overlap). ***[1.75 points]***

Attention

1. The wave properties of the light are not considered in this problem.
2. Neglect any chromatic dispersion in the fiber.
3. The speed of light in vacuum is c = 2.998×10^8 m/s
4. You may use the following formulae:

• The length of a small arc element ds in the xOz plane is

$$ds = dz\sqrt{1 + \left(\frac{dx}{dz}\right)^2}$$



• $$\int \frac{dx}{\sqrt{a^2 - b^2x^2}} = \frac{1}{b} Arc\,\boldsymbol{sin}\frac{bx}{a}$$

• $$\int \frac{x^2dx}{\sqrt{a^2 - b^2x^2}} = -\frac{x\sqrt{a^2 - b^2x^2}}{2b^2} + \frac{a^2 Arc\,\boldsymbol{sin}\frac{bx}{a}}{2b^3}$$

• $\boldsymbol{Arc\,sin}\,x$ is the inverse function of the sine function. Its value equals the less angle the sine of which is x. In other words, if $y = \boldsymbol{Arc\,sin}\,x$ then $\boldsymbol{sin}\,y = x$.