.

Solution 1

(a) $m\ddot{X}_n = S(X_{n+1} - X_n) - S(X_n - X_{n-1})$.	0.7
(b) Let $X_n = A \sin nka \cos(\omega t + \alpha)$, which has a harmonic time dependence. By analogy with the spring, the acceleration is $\ddot{X}_n = -\omega^2 X_n$.	
Substitute into (a): $-mA\omega^2 \sin nka = AS\{\sin(n+1)ka - 2\sin nka + \sin(n-1)ka\}$	
$= -4SA \sin nka \sin^2 \_ka$.	0.6
Hence $\omega^2 = (4S/m)\sin^2 \_ka$.	0.2
To determine the allowed values of k, use the boundary condition $\sin(N+1)ka = \sin kL = 0$.	0.7
The allowed wave numbers are given by $kL = \pi, 2\pi, 3\pi, \ldots, N\pi$ (N in all),	0.3
and their corresponding frequencies can be computed from $\omega = \omega_0 \sin \_ka$,	
in which $\omega_{max} = \omega_0 = 2(S/m)^{–}$ is the maximum allowed frequency.	0.4
(c) $\langle E(\omega)\rangle = \dfrac{\sum_{p=0}^{\infty} p\hbar\omega P_p(\omega)}{\sum_{p=0}^{\infty} P_p(\omega)}$	
First method: $\dfrac{\sum_{n=0}^{\infty} n\hbar\omega e^{-n\hbar\omega/k_BT}}{\sum_{n=0}^{\infty} e^{-n\hbar\omega/k_BT}} = k_BT^2 \dfrac{\partial}{\partial T} \ln \sum_{n=0}^{\infty} e^{-n\hbar\omega/k_BT}$	1.5
The sum is a geometric series and is $\{1 - e^{-\hbar\omega/k_BT}\}^{-1}$	0.5
We find $\langle E(\omega)\rangle = \dfrac{\hbar\omega}{e^{\hbar\omega/k_BT} - 1}$.	
Alternatively: denominator is a geometric series = $\{1 - e^{-\hbar\omega/k_BT}\}^{-1}$	(0.5)
Numerator is k_BT^2(d/dT) (denominator) = $e^{-\hbar\omega/k_BT}\{1 - e^{-\hbar\omega/k_BT}\}^{-2}$ and result follows.	(1.5)

A non-calculus method:
Let $D = 1 + e^{-x} + e^{-2x} + e^{-3x} + \ldots$, where $x = \hbar\omega/k_BT$. This is a geometric series and equals $D = 1/(1 - e^{-x})$. Let $N = e^{-x} + 2e^{-2x} + 3e^{-3x} + \ldots$. The result we want is N/D. Observe

(2.0)

$$\begin{aligned} D - 1 &= e^{-x} + e^{-2x} + e^{-3x} + e^{-4x} + e^{-5x} + \ldots\ldots \\ (D-1)e^{-x} &= \qquad\quad e^{-2x} + e^{-3x} + e^{-4x} + e^{-5x} + \ldots\ldots \\ (D-1)e^{-2x} &= \qquad\qquad\qquad e^{-3x} + e^{-4x} + e^{-5x} + \ldots \end{aligned}$$

Hence $N = (D - 1)D$ or $N/D = D - 1 = \dfrac{e^{-x}}{1 - e^{-x}} = \dfrac{1}{e^x - 1}$.

(d) From part (b), the allowed k values are $\pi/L, 2\pi/L, \ldots, N\pi/L$.

Hence the spacing between allowed k values is π/L, so there are $(L/\pi)\Delta k$ allowed modes in the wave-number interval Δk (assuming $\Delta k >> \pi/L$). 1.0

(e) Since the allowed k are $\pi/L, \ldots, N\pi/L$, there are N modes. 0.5

Follow the problem: 0.5
$d\omega/dk = \_a\omega_0 \cos \_ka$ from part (a) & (b)

$= \frac{1}{2}a\sqrt{\omega^2_{\max} - \omega^2}$, $\omega_{\max} = \omega_0$. This second form is more convenient for integration.

The number of modes dn in the interval $d\omega$ is

$$dn = (L/\pi)\Delta k = (L/\pi)\,(dk/d\omega)\,d\omega$$

0.5 for eit

$$= (L/\pi)\{\_\,a\omega_0 \cos \_\,ka\}^{-1}\,d\omega$$

$$= \frac{L}{\pi}\frac{2}{a}\frac{1}{\sqrt{\omega^2_{\max} - \omega^2}}d\omega$$

$$= \frac{2(N+1)}{\pi}\frac{1}{\sqrt{\omega^2_{\max} - \omega^2}}d\omega$$

This part is necessary f E_T below,

but not for number of modes

Total number of modes $= \int dn = \int_0^{\omega_{\max}} \dfrac{2(N+1)}{\pi}\dfrac{d\omega}{\sqrt{\omega^2_{\max} - \omega^2}} = N + 1 \approx N$ for large N. (0.5)

Total crystal energy from (c) and dn of part (e) is given by

$$E_T = \frac{2N}{\pi}\int_0^{\omega_{\max}} \frac{\hbar\omega}{e^{\hbar\omega/k_BT} - 1}\frac{d\omega}{\sqrt{\omega^2_{\max} - \omega^2}}.$$

0.7

(f) Observe first from the last formula that E_T increases monotonically with temperature since

$\{e^{\hbar\omega/kT} - 1\}^{-1}$ is increasing with T. 0.2

When $T \to 0$, the term – 1 in the last result may be neglected in the denominator so 0.2

$$E_T \underset{T\to 0}{\approx} \frac{2N}{\pi}\int \hbar\omega\, e^{-\hbar\omega/k_BT}\,\frac{1}{\sqrt{\omega_{\max}^2-\omega^2}}\,d\omega$$ 0.3

$$= \frac{2N}{\hbar\pi\omega_{\max}}(k_BT)^2\int_0^\infty \frac{xe^{-x}}{\sqrt{1-(k_BTx/\hbar\omega_{\max})^2}}dx$$ 0.2

which is quadratic in T (denominator in integral is effectively unity) hence C_V is linear in T near absolute zero. 0.2

Alternatively, if the summation is retained, we have

$$E_T = \frac{2N}{\pi}\sum_\omega \frac{\hbar\omega}{e^{\hbar\omega/k_BT}-1}\frac{\Delta\omega}{\sqrt{\omega_{\max}^2-\omega^2}} \to_{T\to 0} \frac{2N}{\pi}\sum_\omega \hbar\omega e^{-\hbar\omega/k_BT}\,\frac{\Delta\omega}{\sqrt{\omega_{\max}^2-\omega^2}}$$

$$= \frac{2N}{\pi}\frac{(k_BT)^2}{\hbar\omega}\sum_y e^{-y}y\Delta y$$ (0.5)

When $T\to\infty$, use $e^x \approx 1 + x$ in the denominator, 0.2

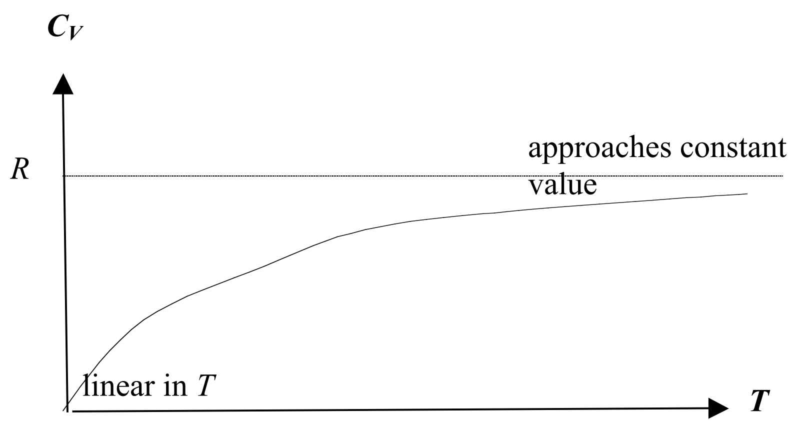
$$E_T \underset{T\to\infty}{\approx} \frac{2N}{\pi}\int_0^{\omega_{\max}} \frac{\hbar\omega}{\hbar\omega/k_BT}\frac{1}{\sqrt{\omega_{\max}^2-\omega^2}}d\omega = \frac{2N}{\pi}k_BT\frac{\pi}{2},$$ 0.1

which is linear; hence $C_V \to Nk_B = R$, the universal gas constant. This is the Dulong-Petit rule.

Alternatively, if the summation is retained, write denominator as $e^{\hbar\omega/k_BT} - 1 \approx \hbar\omega/k_BT$ and (0.2)

$E_T \to_{T\to\infty} \frac{2N}{\pi}k_BT\sum_\omega \frac{\Delta\omega}{\sqrt{\omega_{\max}^2-\omega^2}}$ which is linear in T, so C_V is constant.

Sketch of C_V versus T: 0.5



Answer sheet: Question 1

(a) Equation of motion of the n^{th} mass is:

$$m\ddot{X}_n = S(X_{n+1} - X_n) - S(X_n - X_{n-1}).$$

(b) Angular frequencies ω of the chain's vibration modes are given by the equation:

$\omega^2 = (4S/m) \sin^2$ _*ka*.

Maximum value of ω is: $\omega_{max} = \omega_0 = 2(S/m)^{-}$

The allowed values of the wave number k are given by:

$$\pi/L,\ 2\pi/L,\ \ldots,\ N\pi/L.$$

How many such values of k are there? *N*

(f) The average energy per frequency mode ω of the crystal is given by:

$$\langle E(\omega)\rangle = \frac{\hbar\omega}{e^{\hbar\omega/k_B T}-1}$$

(g) There are how many allowed modes in a wave number interval Δk?

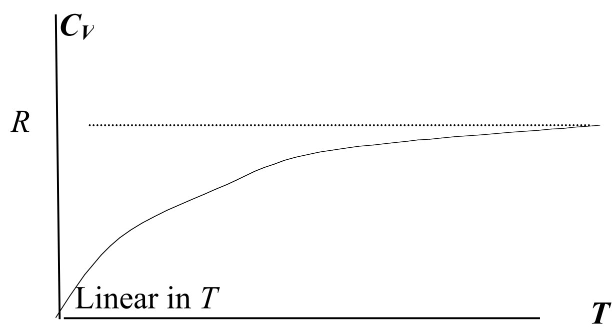
$$(L/\pi)\Delta k.$$

(e) The total number of modes in the lattice is: N

Total energy E_T of crystal is given by the formula:

$$E_T = \frac{2N}{\pi}\int_0^{\omega_{max}} \frac{\hbar\omega}{e^{\hbar\omega/k_B T}-1}\frac{d\omega}{\sqrt{\omega_{max}^2-\omega^2}}.$$

(h) A sketch (graph) of C_V versus absolute temperature T is shown below.



For $T << 1$, C_V displays the following behaviour: C_V is linear in T.

As $T \to \infty$, C_V displays the following behaviour: $C_V \to Nk_B = R$, the universal gas constant.

Solution to Question 2: The Rail Gun

<u>Proper Solution (taking induced emf into consideration)</u>: (a) Let I be the current supplied by the battery in the absence of back emf. Let i be the induced current by back emf ε_b . Since $\varepsilon_b = d\phi / dt = d(BLx)/dt = BLv$, $\therefore i = Blv / R$. Net current, $I_N = I - i = I - BLv / R$. Forces parallel to rail are: Force on rod due to current is $F_c = BLI_N = BL(I - BLv / R) = BLI - B^2 L^2 v / R$. Net force on rod and young man combined is $F_N = F_c - mg \sin\theta$. (1) Newton's law: $F_N = ma = mdv / dt$. (2) Equating (1) and (2), & substituting for F_c & dividing by *m*, we obtain the acceleration $dv / dt = \alpha - v / \tau$, where $\alpha = BIL / m - g \sin\theta$ and $\tau = mR / B^2 L^2$.	 1 0.5 0.5 0.5 0.5	 3

(b)(i) Since initial velocity of rod = 0, and let velocity of rod at time *t* be *v(t)*, we have $$v(t) = v_\infty\left(1 - e^{-t/\tau}\right), \qquad (3)$$ where $v_\infty(\theta) = \alpha\tau = \frac{IR}{BL}\left(1 - \frac{mg}{BLI}\sin\theta\right)$. Let t_s be the total time he spent moving along the rail, and v_s be his velocity when he leaves the rail, i.e. $$v_s = v(t_s) = v_\infty\left(1 - e^{-t_s/\tau}\right). \qquad (4)$$ $$\therefore t_s = -\tau\ln\left(1 - v_s / v_\infty\right) \qquad (5)$$	0.5 0.5 0.5	 1.5

(b) (ii)		
Let t_f be the time in flight:		
$t_f = \dfrac{2v_s \sin\theta}{g}$ (6)	0.5	
He must travel a horizontal distance w during t_f.		
$w = (v_s \cos\theta)t_f$ (7)		
$t_f = \dfrac{w}{v_s \cos\theta} = \dfrac{2v_s \sin\theta}{g}$ (8) (from (6) & (7))	0.5	
From (8), v_s is fixed by the angle θ and the width of the strait w		
$v_s = \sqrt{\dfrac{gw}{\sin 2\theta}}$. (9)		
$\therefore t_s = -\tau \ln\left(1 - \dfrac{1}{v_\infty}\sqrt{\dfrac{gw}{\sin 2\theta}}\right)$, (Substitute (9) in (5))		1.5
And $t_f = \dfrac{2\sin\theta}{g}\sqrt{\dfrac{gw}{\sin 2\theta}} = \sqrt{\dfrac{2w\tan\theta}{g}}$ (Substitute (9) in (8))	0.5	

(c)

Therefore, total time is: $$T = t_s + t_f = -\tau \ln\left(1 - \frac{1}{v_\infty}\sqrt{\frac{gw}{\sin 2\theta}}\right) + \sqrt{\frac{2w\tan\theta}{g}}$$

The values of the parameters are: B=10.0 T, I= 2424 A, L=2.00 m, R=1.0 Ω, g=10 m/s^2, m=80 kg, and w=1000 m.

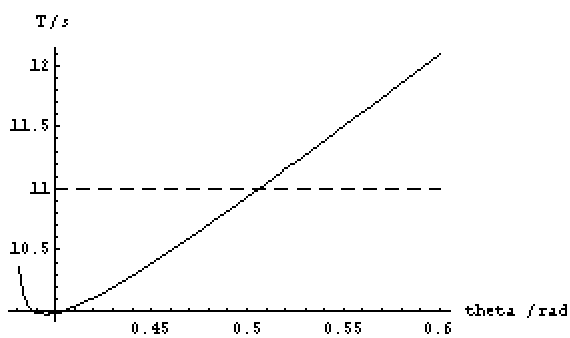
Then $\tau = \frac{mR}{B^2L^2} = \frac{(80)(1.0)}{(10.0)^2(2.00)^2} = 0.20$ s.

$$v_\infty(\theta) = \frac{2424}{(10.0)(2.00)}\left(1 - \frac{(80)(10)}{(10.0)(2.00)(2424)}\sin\theta\right)$$
$$= 121(1 - 0.0165\sin\theta)$$

So,

$$T = t_s + t_f = -0.20\ln\left(1 - \frac{100}{v_\infty}\frac{1}{\sqrt{\sin 2\theta}}\right) + 14.14\sqrt{\tan\theta}$$

By plotting *T* as a function of *θ*, we obtain the following graph:



Note that the lower bound for the range of θ to plot may be determined by the condition $v_s / v_\infty < 1$ (or the argument of ln is positive), and since mg/BLI is small (0.0165), $v_\infty \approx IR/BL$ (= 121 m/s), we have the condition $\sin(2\theta) > 0.68$, i.e. $\theta > 0.37$. So one may start plotting from $\theta = 0.38$.

From the graph, for θ within the range (~0.38, 0.505) radian the time *T* is within 11 s.

Labeling:
0.1 each axis

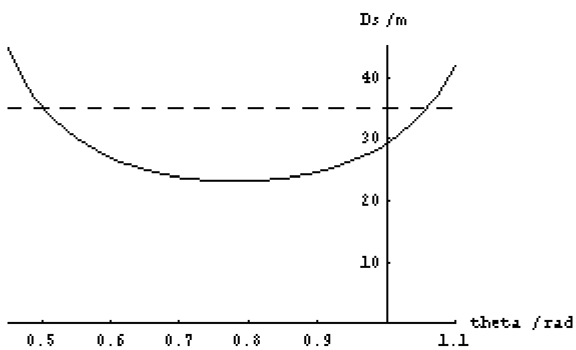
Unit:
0.1 each axis

Proper Range in θ:
0.3 lower limit (more than 0.37, less than 0.5),
0.2 upper limit (more than 0.5 and less than 0.6)

Proper shape of curve: 0.2

Accurate intersection at $\theta = 0.5$: 0.4

1.5

(d) However, there is another constraint, i.e. the length of rail *D*. Let D_s be the distance travelled during the time interval t_s $D_s = \int_0^{t_s} v(t)dt = v_\infty \int_0^{t_s} \left(1 - e^{-t/\tau}\right)dt = v_\infty \left(t + \tau e^{-\beta t}\right)\Big\vert_0^{s} = v_\infty \left[t_s - \tau\left(1 - e^{-\beta t}\right)\right] = v_\infty t_s - v(t_s)\tau$		
i.e. $D_s = -\tau\left[v_\infty(\theta)\ln\left(1 - \frac{1}{v_\infty(\theta)}\sqrt{\frac{gw}{\sin 2\theta}}\right) + \sqrt{\frac{gw}{\sin 2\theta}}\right]$	0.5	
The graph below shows D_s as a function of θ.  It is necessary that $D_s \le D$, which means θ must range between .5 and1.06 radians.	Labeling: 0.1 each axis Unit: 0.1 each axis Proper Range in θ: 0.3 lower limit (more than 0.4, less than 0.49), 0.2 upper limit (more than 0.51 and less than 1.1) Proper shape of curve: 0.2 Accurate intersection at θ = 0.5: 0.4	
In order to satisfy both conditions, θ must range between 0.5 & 0.505 radians. (Remarks: Using the formula for t_f , t_s & D, we get At θ = 0.507, t_f =10.540, t_s = 0.466, giving T = 11.01 s, & D = 34.3 m At θ = 0.506, t_f =10.527, t_s = 0.467, giving T = 10.99 s, & D = 34.4 m At θ = 0.502, t_f =10.478, t_s = 0.472, giving T = 10.95 s, & D =34.96 m At θ = 0.50, t_f =10.453, t_s = 0.474, giving T = 10.93 s, & D =35.2 m, So the more precise angle range is between 0.502 to 0.507, but students are not expected to give such answers. To 2 sig fig T = 11 s. Range is 0.50 to 0.51 (in degree: 28.6^0 to 29.2^0 or 29^0)	0.5	2.5

Alternate Solution (Not taking induced emf into consideration):		
If induced emf is not taken into account, there is no induced current, so the net force acting on the combined mass of the young man and rod is		
$$F_N = BIL - mg\sin\theta \ .$$	0.2 *BIL* 0.2 *mg* $\sin\theta$	
And we have instead $$dv/dt = \alpha,$$ where $$\alpha = BIL/m - g\sin\theta \ .$$		
$$\therefore v(t) = \alpha t$$	0.1	
and $$\therefore v_s = v(t_s) = \alpha t_s$$	0.2	
$$t_f = \frac{2v_s \sin\theta}{g} = \frac{2\alpha t_s \sin\theta}{g} \ .$$ Therefore, $$w = (v_s\cos\theta)t_f = \frac{\alpha^2 t_s^{\ 2}\sin 2\theta}{g} \ ,$$ giving		
$$t_s = \frac{1}{\alpha}\sqrt{\frac{gw}{\sin 2\theta}}$$	0.5	
and $$t_f = \sqrt{\frac{2w\tan\theta}{g}} \ .$$	0.5	
Hence, $$T = t_s + t_f = \frac{1}{\alpha}\sqrt{\frac{gw}{\sin 2\theta}} + \sqrt{\frac{2w\tan\theta}{g}} = \frac{\sqrt{wg}}{\alpha}\frac{\left[1 + 2\left(\frac{\alpha}{g}\right)\sin\theta\right]}{\sqrt{\sin 2\theta}} \ .$$ where $\alpha = BIL/m - g\sin\theta$.		
The values of the parameters are: B=10.0 T, I= 2424 A, L=2.00m, R=1.0 Ω, g=10 m/s^2, m=80 kg, and w=1000 m. Then,		
$$T = \frac{100}{\alpha}\frac{\left[1 + 0.20\alpha\sin\theta\right]}{\sqrt{\sin 2\theta}}$$ where $\alpha = 606 - 10\sin\theta$.	0.3	2

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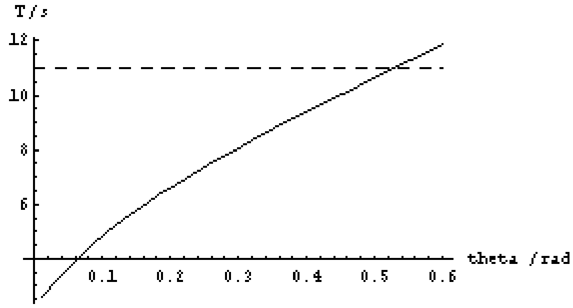
<td>

For θ within the range (~0, 0.52) radian the time T is within 11 s.</td>
<td>Labeling:
0.1 each axis

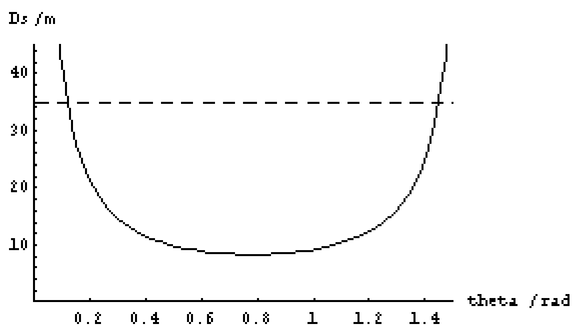
Unit:
0.1 each axis

Proper Range in θ:
0.1 lower limit
(more than 0,
less than 0.5),
0.2 upper limit
(more than 0.52
and less than 0.8)

Proper shape of curve: 0.2

Accurate intersection at $\theta = 0.52$: 0.4</td>
<td>1.3</td>
</tr>
<tr>
<td>However, there is another constraint, i.e. the length of rail D.
Let D_s be the distance travelled during the time interval t_s

$$D_s = \frac{gw}{2\alpha \sin 2\theta} = \frac{5000}{\alpha \sin 2\theta}$$
which is plotted below



It is necessary that $D_s \le D$, which means θ must range between 0.11 and 1.43 radians.</td>
<td>Labeling:
0.1 each axis

Unit:
0.1 each axis

Proper Range in θ:
0.1 lower limit
(more than 0.08,
less than 0.11),
0.1 upper limit
(more than 0.52
and less than 1.5)

Proper shape of curve: 0.2

Accurate intersection at $\theta = 0.11$: 0.4</td>
<td>1.2</td>
</tr>
<tr>
<td>In order to satisfy both conditions, θ must range between 0.11 & 0.52 radians.</td>
<td></td>
<td>0.5</td>
</tr>
</table>

Question 3 - Marking Scheme

(a) Since $W(v) = 4\pi \left(\frac{M}{2\,\pi\,R\,T} \right)^{3/2} v^2 \; e^{-M\,v^2/(2RT)}$,

$$\bar{v} = \int_0^\infty v \; W(v)\, dv =$$

$$= \int_0^\infty v \;\; 4\pi \left(\frac{M}{2\,\pi\,R\,T} \right)^{3/2} v^2 \; e^{-M\,v^2/(2RT)} \; dv$$

$$= \int_0^\infty 4\pi \left(\frac{M}{2\,\pi\,R\,T} \right)^{3/2} v^3 \; e^{-M\,v^2/(2RT)} \; dv$$

$$= 4\pi \left(\frac{M}{2\,\pi\,R\,T} \right)^{3/2} \int_0^\infty v^3 \; e^{-M\,v^2/(2RT)} \; dv$$

$$= 4\pi \left(\frac{M}{2\,\pi\,R\,T} \right)^{3/2} \frac{4\,R^2\,T^2}{2\,M^2}$$

$$= \sqrt{\frac{8\,R\,T}{\pi\,M}}$$

Marking Scheme:

Performing the integration correctly:	1 mark
Simplifying	0.5 marks

Subtotal for the section 1.5 marks

(b) Assuming an ideal gas, $P V = N k T$, so that the concentration of the gas molecules, *n*, is given by

$$n = \frac{N}{V} = \frac{P}{k T}$$

the impingement rate is given by

$$J = \frac{1}{4} n \bar{v}$$

$$= \frac{1}{4} \frac{P}{k T} \sqrt{\frac{8 R T}{\pi M}}$$

$$= P \sqrt{\frac{8 R T}{16 k^2 T^2 \pi M}}$$

$$= P \sqrt{\frac{N_A k}{2 k^2 T \pi M}}$$

$$= P \sqrt{\frac{1}{2 k T \pi m}}$$

$$= \frac{P}{\sqrt{2 \pi m k T}}$$

where we have note that $R = N_A k$ and $m = \frac{M}{N_A}$ (N_A being Avogadro number).

Marking Scheme:

Using ideal gas formula to estimate concentration of gas molecules: 0.7 marks
Simplifying expression: 0.4 marks
Using $R = N k$, and the formula for *m*; (0.2 mark each) 0.4 marks

Subtotal for the section 1.5 marks

(c) Assuming close packing, there are approximately 4 molecules in an area of 16 r^2 m^2. Thus, the number of molecules in 1 m^2 is given by

$$n_1 = \frac{4}{16\left(3.6 \times 10^{-10}\right)^2} = 1.9 \times 10^{18} \text{ m}^{-2}$$

However at (273 + 300) K and 133 Pa, the impingement rate for oxygen is

$$J = \frac{P}{\sqrt{2\pi mkT}}$$

$$= \frac{133}{\sqrt{2\pi\left(\frac{32 \times 10^{-3}}{6.02 \times 10^{23}}\right)(1.38 \times 10^{-23})573}}$$

$$= 2.6 \times 10^{24} \text{ m}^{-2} \text{ s}^{-1}$$

Therefore, the time needed for the deposition is $\frac{\boldsymbol{n}_1}{\boldsymbol{J}} = 0.7\ \mu\text{s}$

The calculated time is too short compared with the actual processing.

Marking Scheme:

Estimation of number of molecules in 1 m^2 :	0.4 marks
Calculation the impingement rate:	0.6 marks
Taking note of temperature in Kelvin	0.3 marks
Calculating the time	0.4 marks

Subtotal for the section 1.7 marks

(d) With activation energy of 1 eV and letting the velocity of the oxygen molecule at this energy is v_1, we have

$$\frac{1}{2} m v_1^2 = 1.6\times 10^{-19} \text{ J}$$

$$\Rightarrow v_1 = 2453.57 \quad \text{ms}^{-1}$$

At a temperature of 573 K, the distribution of the gas molecules is

We can estimate the fraction of the molecules with speed greater than 2454 ms^{-1} using the trapezium rule (or any numerical techniques) with ordinates at 2453, 2453 + 500, 2453 +1000. The values are as follows:

Velocity, v	Probability, $W(v)$
2453	1.373×10^{-10}
2953	2.256×10^{-14}
3453	6.518×10^{-19}

Using trapezium rule, the fraction of molecules with speed greater than 2453 ms^{-1} is given by

$$\text{fraction of molecules} = \frac{500}{2}\left[\left(1.373\times 10^{-10}\right)+\left(2 \times 2.256 \times 10^{-14}\right)+\left(6.518 \times 10^{-19}\right)\right]$$

$$f = 3.43\times 10^{-8}$$

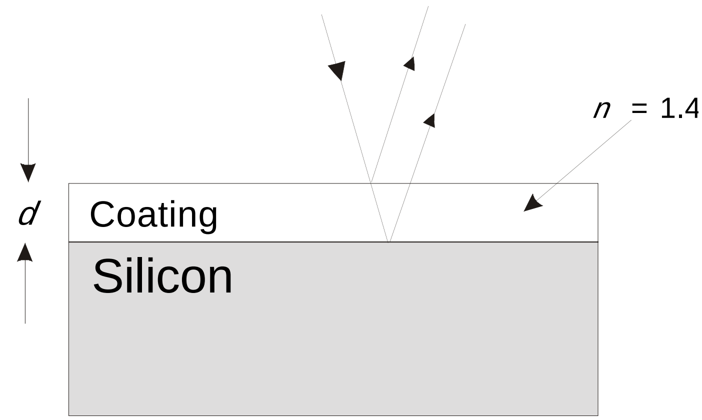
Thus the time needed for the deposition is given by 0.7 µs/(3.43 x 10^{-8}) that is 20.4 s

Marking Scheme

Computing the value of the cut-off energy or velocity: 0.6 marks
Estimating the fraction of molecules 1.2 marks
Correct method of final time 0.4 marks
Correct value of final time 0.6 marks

Subtotal for the section 2.8 marks

(e) For destructive interference, optical path difference = $2\,d = \dfrac{\lambda'}{2}$ where $\lambda' = \dfrac{\lambda_{\text{air}}}{n}$ is the wavelength in the coating.



The relation is given by:

$$d = \frac{\lambda_{\text{air}}}{4\,n}$$

Plugging in the given values, one gets d = 105 or 105.2 nm.

Derive equation:

Finding the optical path length 0.2 marks
Knowing that there is a phase change at the reflection 0.5 marks
Putting everything together to get the final expression 0.6 marks
Subtotal: 1.3 marks

Computation of d: 0.6 marks
Getting the correct number of significant figures: 0.6 marks
Subtotal: 1.2 marks

Subtotal for Section **2.5 marks**

TOTAL **10 marks**