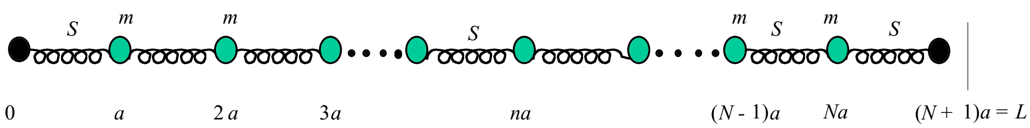
3rd Asian Physics Olympiad
Singapore
Theoretical Competition
Wednesday, 8 May, 2002

Please read this first:

1. The examination lasts for 5 hours. There are 3 questions.
2. Use only the pen issued.
3. Use only the front side of the sheets of paper provided. Do not use the side marked with a cross.
4. Each question should be answered on separate sheets of paper.
5. For each question, in addition to the ***blank writing sheets*** on which you may write, there is an ***answer sheet*** where you must summarize the results you have obtained. **Numerical results should be written with as many digits as are appropriate to the given data. Do not forget to state the units.**
6. Write on the blank sheets of paper whatever you consider is required for the solution of the questions and that you wish to be marked. However you should use mainly equations, numbers, symbols and diagrams. Please use as little text as possible.
7. It is absolutely essential that you enter your ***Country*** and your student number (***Student No.***) at the top of each sheet of paper used. In addition, on the blank sheets of paper used for each question, you should enter the number of the question (***Question No.***), the progressive number of each sheet (***Page No.***) and the total number of blank sheets that you have used and wish to be marked for each question (***Total pages***). It is also helpful to write the question number and the section label of the part you are answering at the beginning of each sheet of writing paper. If you use some blank sheets of paper for notes that you do not wish to be marked, put a large cross through the whole sheet and do not include it in your numbering.
8. When you have finished, arrange all sheets in proper order (for each question put answer sheets first, then used sheets in order, followed by the sheets you do not wish to be marked. Put unused sheets and the printed question at the bottom). Place the papers for each question inside the envelope labeled with the appropriate question number, and leave everything on your desk. You are not allowed to take any sheet of paper out of the room.

Theoretical Question 1 (vibrations of a linear crystal lattice)

A very large number N of movable identical point particles ($N >> 1$), each with mass m, are set in a straight chain with $N + 1$ identical massless springs, each with stiffness (spring constant) S, linking them to each other and the ends attached to two additional immovable particles. See figure. This chain will serve as a model of the vibration modes of a one-dimensional crystal. When the chain is set in motion, the longitudinal vibrations of the chain can be looked upon as a superposition of simple oscillations (called modes) each with its own characteristic mode frequency.



(a) Write down the equation of motion of the n^{th} particle. [0.7 marks]

(b) To attempt to solve the equation of motion of part (a) use the trial solution

$$X_n(\omega) = A \sin nka \cos(\omega t + \alpha),$$

where $X_n(\omega)$ is the displacement of the n^{th} particle from equilibrium, ω the angular frequency of the vibration mode and A, k and α are constants; k and ω are the <u>wave numbers</u> and <u>mode frequencies</u> respectively. For each k, there will be a corresponding frequency ω. Find the dependence of ω on k, the allowed values of k, and the maximum value of ω. The chain's vibration is thus a superposition of all these vibration modes. Useful formulas:

$(d/dx) \cos \alpha x = - \alpha \sin \alpha x,$ $(d/dx) \sin \alpha x = \alpha \cos \alpha x,\ \alpha = \text{constant}.$

$\sin(A + B) = \sin A \cos B + \cos A \sin B,$ $\cos(A + B) = \cos A \cos B - \sin A \sin B$

[2.2 marks]

According to Planck the energy of a photon with a frequency of ω is $\hbar\omega$, where $\hbar$ is the Planck constant divided by 2π. Einstein made a leap from this by assuming that a given crystal vibration mode with frequency ω also has this energy. Note that a vibration mode is not a particle, but a simple oscillation configuration of the entire chain. This vibration mode is analogous to the photon and is called a *phonon*. We will follow up the consequences of this idea in the rest of the problem. Suppose a crystal is made up of a very large ($\sim 10^{23}$) number of particles in a straight chain.

(c) For a given allowed ω (or k) there may be no phonons; or there may be one; or two; or any number of phonons. Hence it makes sense to try to calculate the *average energy* $\langle E(\omega)\rangle$ of a *particular* mode with a frequency ω. Let $P_p(\omega)$ represent the probability that there are p phonons with this frequency ω. Then the required average is

$$\langle E(\omega)\rangle = \frac{\sum_{p=0}^{\infty} p\hbar\omega P_p(\omega)}{\sum_{p=0}^{\infty} P_p(\omega)}.$$

Although the phonons are discrete, the fact that there are so many of them (and the P_p becomes tiny for large p) allows us to extend the sum to $p = \infty$, with negligible error. Now the probability P_p is given by Boltzmann's formula

$$P_p(\omega) \propto \exp(-p\hbar\omega/k_B T),$$

where k_B is Boltzmann's constant and T is the absolute temperature of the crystal, assumed constant. The constant of proportionality does not depend on p. Calculate the average energy for phonons of frequency ω. Possibly useful formula: $(d/dx)\, e^{f(x)} = (df/dx)\, e^{f(x)}$.

[2 marks]

(d) We would like next to compute the *total* energy E_T of the crystal. In part (c) we found the average energy $\langle E(\omega)\rangle$ for the vibration mode ω. To find E_T we must multiply $\langle E(\omega)\rangle$ by the number of modes of the crystal per unit of frequency ω and then sum up all these for the entire range from $\omega = 0$ to ω_{max}. Take an interval Δk in the range of wave numbers. For very large N and for Δk much larger than the spacing between successive (allowed) k values, how many modes can be found in the interval Δk?

[1 mark]

(e) To make use of the results of (a) and (b), approximate Δk by $(dk/d\omega)d\omega$ and replace any sum by an integral over ω. (It is more convenient to use the variable ω in place of k at this point.) State the total number of modes of the crystal in this approximation. Also derive an expression E_T but do not evaluate it. The following integral may be useful: $\int_0^1 dx/\sqrt{1-x^2} = \pi/2$. [2.2 marks]

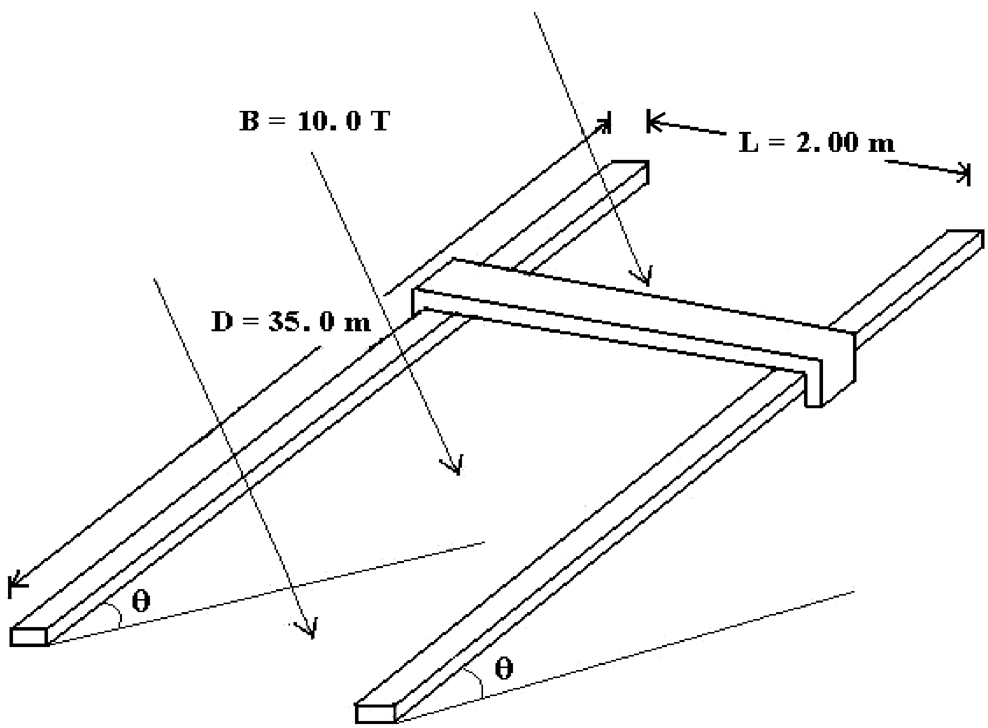
(f) The molar heat capacity C_V of a crystal at constant volume is experimentally accessible: $C_V = dE_T/dT$ (T = absolute temperature). For the crystal under discussion determine the dependence of C_V on T for very large and very low temperatures (i.e., is it constant, linear or power dependent for an interval of the temperature?). Sketch a qualitative graph of C_V versus T, indicating the trends predicted for very low and very high T.

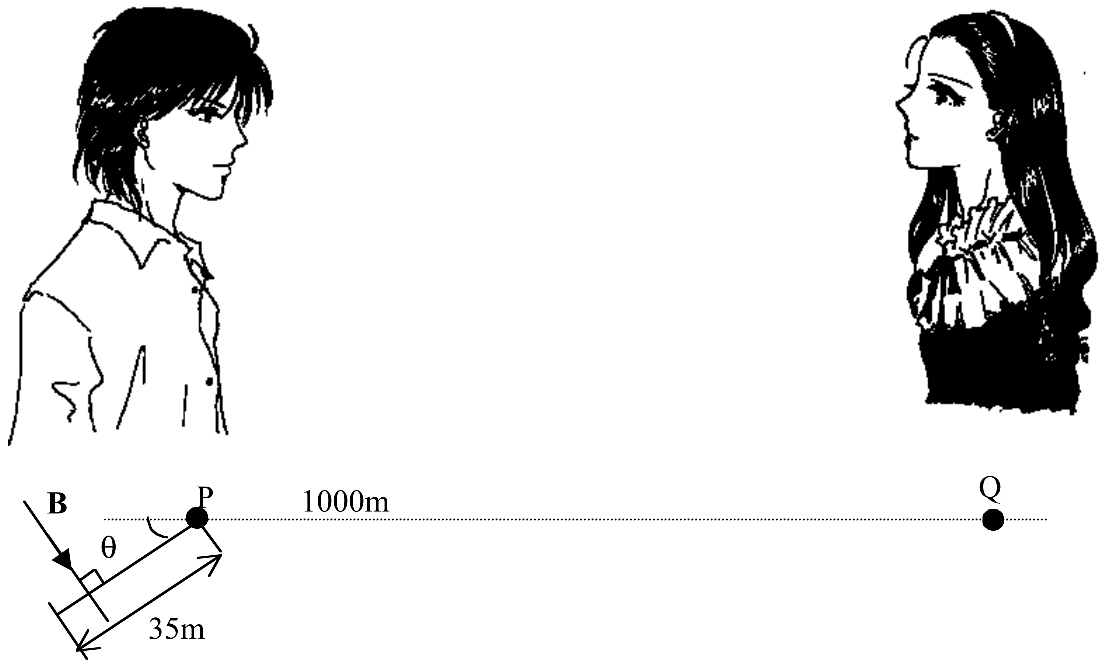
[1.9 marks]

Theoretical Question 2 (the rail gun)

A young man at P and a young lady at Q were deeply in love. These two places are separated by a strait of width w = 1000 m. After learning about the theory of rail gun in class, the young man could not wait to construct such a device to launch himself across the strait. He constructed a ramp of adjustable elevation of angle θ on which he laid two metal rails (the length of each rail is D = 35.0 m) in parallel, separated by L = 2.00 m. He managed to connect a 2424 V DC power supply to the ends of the rails. A conducting bar can slide freely on the metal rails such that he could hang on to it safely as it slides.

A skilled engineer, moved by all these efforts, designed a system that can produce a B = 10.0 T magnetic field that can be directed perpendicular to the plane of the rails. The mass of the young man is 70 kg. The mass of the conducting bar is 10 kg and its resistance is R = 1.0 Ω.





Just after he had completed the construction and checked that it worked perfectly, he received a call from the young lady, sobbing and telling him that her father was going to marry her off to a rich man unless he can arrive at Q within 11 seconds after the call, and having said that she hang up.

The young man immediately got into action and launched himself across the strait to Q.

Show, using the steps listed below, whether it is possible for him to make it in time, and if so, what is the range of θ he must set the ramp?

(a) Derive an expression for the acceleration of the young man parallel to the rail. [3 marks]

(b) Obtain an expression in terms of θ for the time spent

i. on the rails, t_s and

ii. in flight, t_f.

[3 marks]

(c) Plot a graph of the total time $T = t_s + t_f$ against the angle of inclination θ. [1.5 marks]

(d) By considering the relevant parameters of this device, obtain the range of angles that he should set. Plot another graph if necessary. [2.5 marks]

Make the following assumptions:

1) The time between the end of the call and all preparations (such as setting θ to the appropriate angle) for the launch is negligible. This is to say, the launch is considered to start at time $t = 0$ when the bar (with the young man hanging to it) is starting to move.
2) The young man may start his motion from any point along the metal rails.

3) The higher end of the ramp and Q is at the same level, and the distance between them is $w = 1000$ m.
4) There is no question about safety such as when landing, electric shocks, etc.
5) The resistance of the metal rails, the internal resistance of the power supply, the friction between the conducting bar and the rails and the air resistance are all negligible.
6) Take acceleration due to gravity as $g = 10$ m/s^2.

Some Mathematical notes:

1. $\int e^{-ax} dx = -\frac{e^{-ax}}{a}$.

2. The solution to $\frac{dx}{dt} = a + bx$ is given by

$$x(t) = \frac{a}{b}(e^{bt} - 1) + x(0)e^{bt} .$$

Theoretical Question 3 (wafer fabrication)

Wafer fabrication refers to the production of semiconductor chips from silicon. In modern technologies there are more than 20 processes; we are going to concentrate on thin films deposition.

In wafer fabrication process, thin films of various materials are deposited on the surface of the silicon wafer. The surface of the substrate must be extremely clean before the process of deposition. The presence of traces of oxygen or other elements will result in the formation of a contamination layer. The rate of formation of this layer is determined by the impingement rate of the gas molecules hitting the substrate surface. Assuming the number of molecules per unit volume is n, the impingement rate on a unit area of the substrate from the gas is given by

$$J = \frac{1}{4} n \bar{v}$$

where $\bar{v}$ is the average or mean speed of the gas molecules.

(a) Assuming that the gas molecules obey a Maxwell-Boltzmann distribution,

$$W(v) = 4\pi \left(\frac{M}{2 \pi R T} \right)^{3/2} v^2 \; e^{-M v^2 / (2RT)} ,$$

where $W(v)\,dv$ is the fraction of molecules whose speed lie between v and $v + dv$, M is the molar mass of the gas, T is the gas temperature and R is the gas constant, show that the average or mean speed of the gas molecules is given by

$$\bar{v} = \int_0^{\infty} v \; W(v) \; dv = \sqrt{\frac{8 R T}{\partial M}}$$

[1.5 marks]

(b) Assuming that the gases behave as an ideal gas at low pressure, P, show that the rate of impingement is given by

$$J = \frac{P}{\sqrt{2 \pi m k T}}$$

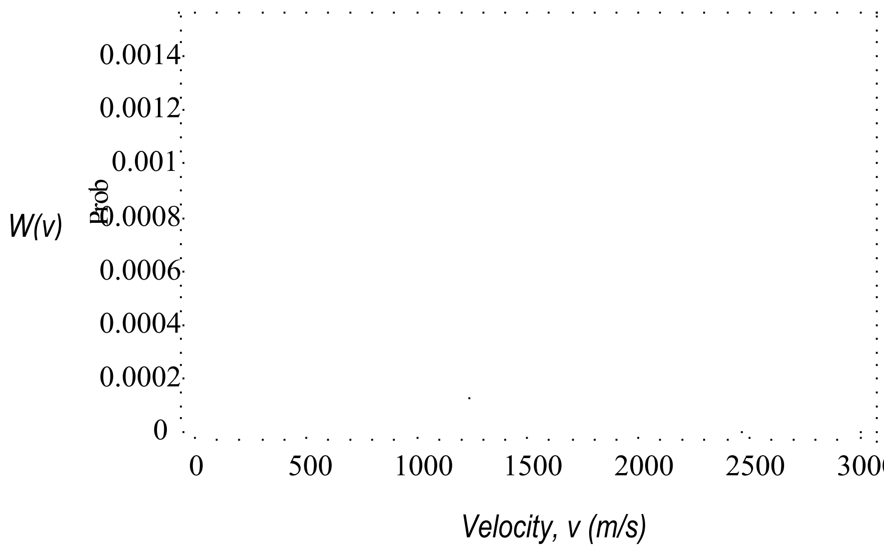
where m is the mass of the molecule and T is the temperature of the gas.

[1.5 marks]

(c) If the residual pressure of oxygen in a vacuum system is 133 Pa, and by modelling the oxygen molecule as a sphere of radius approximately 3.6×10^{-10} m, estimate how long it takes to deposit a molecule-thick layer of oxygen on the wafer at 300° Celsius, assuming that all the oxygen molecules which strike the silicon wafer surface are deposited. Assume also that oxygen molecules in the layer are arranged side by side.

[1.7 marks]

(d) In reality, not all molecules of oxygen react with the silicon. This can be modeled by the concept of activation energy where the reacting molecules should have total energy greater than the activation energy before it can react. Physically this activation energy describes the fact that chemical bonds between the silicon atoms have to be broken before a new bond between silicon and oxygen atoms is formed. Assuming an activation energy for the reaction to be 1 eV, estimate again how long it would take to deposit one atomic layer of oxygen at the above temperature. You may assume that the area under the Maxwell distribution in part (a) is unity.



[2.8 marks]

(e) For lithography processes, the clean silicon wafer is coated evenly with a layer of transparent polymer (photo-resist) of refractive index $\mu = 1.40$. To measure the thickness of this photo-resist, the wafer is illuminated with collimated monochromatic beam of light of wavelength $\lambda = 589$ nm. For a certain minimum thickness of photo-resist, $\boldsymbol{d}$, there is a destructive interference of reflected light, assuming normal incidence on the coating. Derive an expression for relation between d, μ and λ. Calculate d using the given data. In this point you may assume that silicon behaves as a medium with a refractive index greater than 1.40 and you may ignore multiple reflections.

[2.5 marks]

The following data may be helpful:

Molar mass of oxygen is 32 g mol $^{-1}$.
Boltzmann constant, $k = 1.38 \times 10^{-23}$ J K^{-1}.
Avogadro number, $N_A = 6.02 \times 10^{23}$ mol $^{-1}$

Useful formula:

$$\int x^3 \, e^{-k x^2} \, dx = -\frac{1}{2} e^{-k x^2} \left(\frac{1}{k^2} + \frac{x^2}{k} \right)$$

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Answer sheet: ***Theoretical Question 1*** **(vibrations of a linear crystal lattice)**

(a) Equation of motion of the n^{th} particle is:

(b) Angular frequencies ω of the chain's vibration modes are given by the equation:

The allowed values of the wave number k are given by:

Maximum value of ω is:

(c) The average energy for phonons of frequency ω is given by:

(d) There are how many allowed modes in a wave number interval Δk?

(e) The total number of modes in the crystal is:

Total energy E_T of crystal is given by the formula:

(f) A sketch (graph) of C_V versus absolute temperature T is shown below.

For $T << 1$, C_V displays the following dependence:

As $T \to \infty$, C_V displays the following dependence:

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Answer Sheet: Theoretical question 2 **(The Rail Gun)**

a) Expression for the acceleration of the young man parallel to the rail.

$dv/dt =$

b) Obtain an expression for the time spent

a. on the rails t_s.

$t_s =$

b. in flight t_f

$t_f =$

c) (Examiner will refer to your tables of values and graphs in your solution.)

d) Range of angles:

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Answer Sheet: Theoretical Question 3 (**Wafer Fabrication)**

(c) The time needed to deposit **ONE** layer is

(d) The new time needed to deposit **ONE** layer is:

(e) Relation between d, μ and λ :

Value of d: