Theoretical Question 1

When will the Moon become a Synchronous Satellite?

The period of rotation of the Moon about its axis is currently the same as its period of revolution about the Earth so that the same side of the Moon always faces the Earth. The equality of these two periods presumably came about because of actions of tidal forces over the long history of the Earth-Moon system.

However, the period of rotation of the Earth about its axis is currently shorter than the period of revolution of the Moon. As a result, lunar tidal forces continue to act in a way that tends to slow down the rotational speed of the Earth and drive the Moon itself further away from the Earth.

In this question, we are interested in obtaining an estimate of how much more time it will take for the rotational period of the Earth to become equal to the period of revolution of the Moon. The Moon will then become a synchronous satellite, appearing as a fixed object in the sky and visible only to those observers on the side of the Earth facing the Moon. We also want to find out how long it will take for the Earth to complete one rotation when the said two periods are equal.

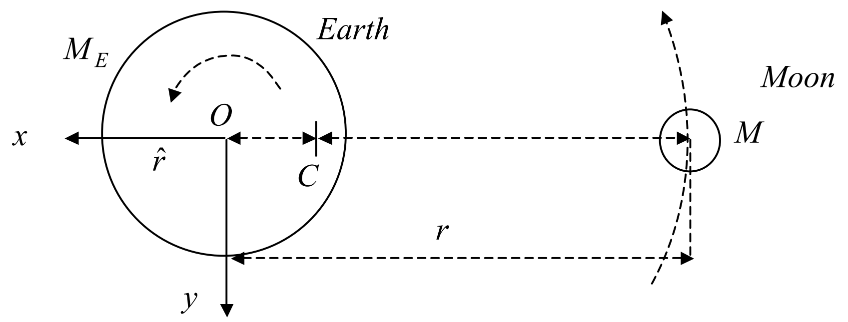
Two right-handed rectangular coordinate systems are adopted as reference frames. The third coordinate axes of these two systems are parallel to each other and normal to the orbital plane of the Moon.

(I)The first frame, called the *CM* frame, is an inertial frame with its origin located at the center of mass C of the Earth-Moon system.

(II)The second frame, called the *xyz* frame, has its origin fixed at the center O of the Earth. Its z-axis coincides with the axis of rotation of the Earth. Its x-axis is along the line connecting the centers of the Moon and the Earth, and points in the direction of the unit vector $\hat{r}$ as shown in Fig.1a. The Moon remains always on the negative x-axis in this frame.

Note that distances in Fig.1a are not drawn to scale. The curved arrows show the directions of the Earth's rotation and the Moon's revolution. The Earth-Moon distance is denoted by r.

Fig. 1a



The following data are given:

(a) At present, the distance between the Moon and the Earth is $r_0 = 3.85\times10^8\,\text{m}$ and increases at a rate of 0.038 m per year.

(b) The period of revolution of the Moon is currently T_M = 27.322 days.

(c) The mass of the Moon is $M = 7.35\times10^{22}\,\text{kg}$.

(d) The radius of the Moon is $R_M = 1.74\times10^6\,\text{m}$.

(e) The period of rotation of the Earth is currently T_E = 23.933 hours.

(f) The mass of the Earth is $M_E = 5.97\times10^{24}\,\text{kg}$.

(g) The radius of the Earth is $R_E = 6.37\times10^6\,m$.

(h) The universal gravitational constant is $G = 6.67259\times10^{-11}\,N\cdot m^2/kg^2$.

The following assumptions may be made when answering questions:

(i) The Earth-Moon system is isolated from the rest of the universe.

(ii) The orbit of the Moon about the Earth is circular.

(iii) The axis of rotation of the Earth is perpendicular to the orbital plane of the Moon.

(iv) If the Moon is absent and the Earth does not rotate, then the mass distribution of the Earth is spherically symmetric and the radius of the Earth is R_E.

(v) For the Earth or the Moon, the moment of inertia I about any axis passing through its center is that of a uniform sphere of mass M and radius R, i.e. $I = \frac{2}{5}MR^2$.

(vi) The water around the Earth is stationary in the *xyz* frame.

Answer the following questions:

(1) With respect to the center of mass C, what is the current value of the total angular momentum L of the Earth-Moon system?

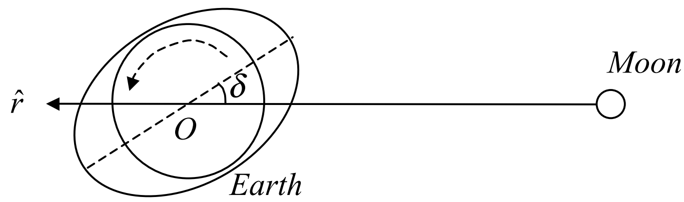
(2) When the period of rotation of the Earth and the period of revolution of the Moon become equal, what is the duration of one rotation of the Earth? Denote the answer as T and express it in units of the present day. Only an approximate solution is required so that iterative methods may be used.

(3) Consider the Earth to be a rotating solid sphere covered with a surface layer of water and assume that, as the Moon moves around the Earth, the water layer is stationary in the *xyz*-frame. In one model, frictional forces between the rotating solid sphere and the water layer are taken into account. The faster spinning solid Earth is assumed to drag lunar tides along so that the line connecting the tidal bulges is at an angle δ with the *x*-axis, as shown in Fig.1b. Consequently, lunar tidal forces acting on the Earth will exert a torque Γ about O to slow down the rotation of the Earth.

The angle δ is assumed to be constant and independent of the Earth-Moon distance r until it vanishes when the Moon's revolution is synchronous with the Earth's rotation so that frictional forces no longer exist. The torque Γ therefore scales with the Earth-Moon distance and is proportional to $\frac{1}{r^6}$.

According to this model, when will the rotation of the Earth and the revolution of the Moon have the same period? Denote the answer as t_f and express it in units of the present year.

Fig.1b



The following mathematical formulae may be useful when answering questions:

(M1) For $0 \le s < r$ and $x = s\cos\theta$:

$$\frac{1}{\sqrt{r^2 + s^2 + 2rx}} \approx \left(\frac{1}{r} - \frac{x}{r^2} + \frac{3x^2 - s^2}{2r^3} + \cdots\right)$$

(M2) If $a \neq 0$ and $\frac{d\omega}{dt} = b\omega^{1-a}$, then $\omega^a(t') - \omega^a(t) = (t' - t)ab$