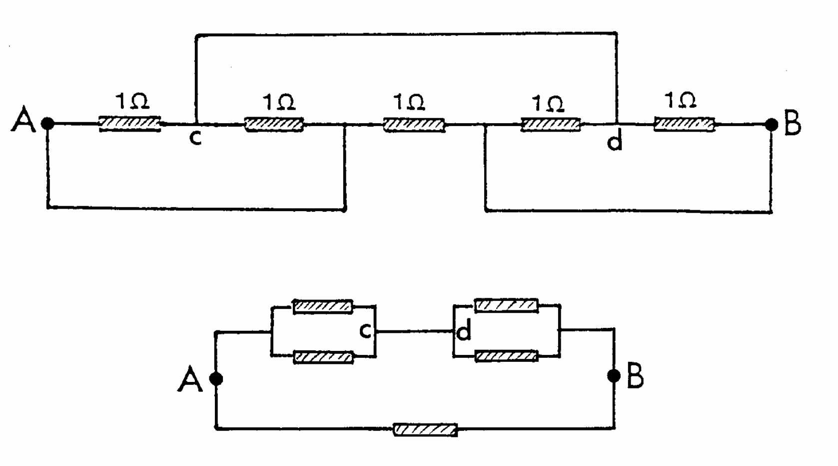
27^{th} INTERNATIONAL PHYSICS OLYMPIAD
OSLO, NORWAY

THEORETICAL COMPETITION
JULY 2 1996

Solution Problem 1

a) The system of resistances can be redrawn as shown in the figure:



The equivalent drawing of the circuit shows that the resistance between point c and point A is 0.5Ω, and the same between point d and point B. The resistance between points A and B thus consists of two connections in parallel: the direct 1Ω connection and a connection consisting of two 0.5Ω resistances in series, in other words two parallel 1Ω connections. This yields

$$R = \underline{\underline{0.5\Omega}}\,.$$

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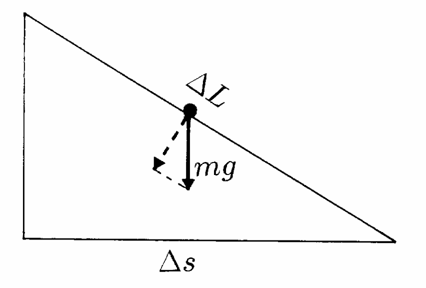
b) For a sufficiently short horizontal displacement Δs the path can be considered straight. If the corresponding length of the path element is ΔL, the friction force is given by

$$\mu\, mg \frac{\Delta s}{\Delta L}$$

and the work done by the friction force equals force times displacement:

$$\mu\ mg\frac{\Delta s}{\Delta L}\cdot\Delta L=\mu\ mg\Delta s.$$



Adding up, we find that along the whole path the total work done by friction forces is $\mu\, mg\, s$. By energy conservation this must equal the decrease *mg h* in potential energy of the skier. Hence

$$h=\underline{\underline{\mu s}}.$$

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c) Let the temperature increase in a small time interval *dt* be *dT*. During this time interval the metal receives an energy *P dt.*

The heat capacity is the ratio between the energy supplied and the temperature increase:

$$C_p=\frac{Pdt}{dT}=\frac{P}{dT/dt}.$$

The experimental results correspond to

$$\frac{dT}{dt}=\frac{T_0}{4}a[1+a(t-t_0)]^{-3/4}=T_0\frac{a}{4}\left(\frac{T_0}{T}\right)^3.$$

Hence

$$C_p=\frac{P}{dT/dt}=\underline{\underline{\frac{4P}{aT_0^{\ 4}}}}T^3.$$

(*Comment*: At low, but not extremely low, temperatures heat capacities of metals follow such a T^3 law.)

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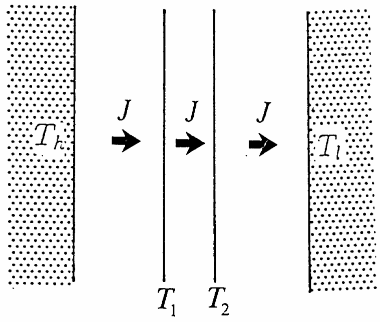
d)



Under stationary conditions the net heat flow is the same everywhere:

$$J = \sigma(T_h^4 - T_1^4)$$
$$J = \sigma(T_1^4 - T_2^4)$$
$$J = \sigma(T_2^4 - T_l^4)$$

Adding these three equations we get

$$3J = \sigma(T_h^4 - T_l^4) = J_0,$$

where J_0 is the heat flow in the absence of the heat shield. Thus $\xi = J/J_0$ takes the value

$$\underline{\underline{\xi = \tfrac{1}{3}}}.$$

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e) The magnetic field can be determined as the superposition of the fields of two *cylindrical* conductors, since the effects of the currents in the area of intersection cancel. Each of the cylindrical conductors must carry a larger current I', determined so that the fraction I of it is carried by the actual cross section (the moon-shaped area).
The ratio between the currents I and I' equals the ratio between the cross section areas:

$$\frac{I}{I'} = \frac{(\frac{\pi}{12} + \frac{\sqrt{3}}{8})D^2}{\frac{\pi}{4}D^2} = \frac{2\pi + 3\sqrt{3}}{6\pi}.$$

Inside one cylindrical conductor carrying a current I' Ampère's law yields at a distance r from the axis an azimuthal field

$$B_\phi = \frac{\mu_0}{2\pi r}\frac{I'\pi r^2}{\frac{\pi}{4}D^2} = \frac{2\mu_0 I' r}{\pi D^2}.$$

The cartesian components of this are

$$B_x = -B_\phi \frac{y}{r} = -\frac{2\mu_0 I' y}{\pi D^2}; \qquad B_y = B_\phi \frac{x}{r} = \frac{2\mu_0 I' x}{\pi D^2}.$$

For the superposed fields, the currents are $\pm I'$ and the corresponding cylinder axes are located at $x = \mp D/4$.

The two x-components add up to zero, while the y-components yield

$$B_y = \frac{2\mu_0}{\pi D^2}[I'(x + D/4) - I'(x - D/4)] = \frac{\mu_0 I'}{\pi D} = \underline{\underline{\frac{6\mu_0 I}{(2\pi + 3\sqrt{3})D}}},$$

i.e., a *constant* field. The direction is along the positive *y*-axis.

Solution problem 2

a) The potential energy gain eV is converted into kinetic energy. Thus

$$\tfrac{1}{2}mv^2 = eV \qquad \text{(non-relativistically)}$$

$$\frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2 = eV \qquad \text{(relativistically).}$$

Hence

$$v = \begin{cases} \sqrt{2eV/m} & \text{(non - relativistically)} \\ c\sqrt{1-\left(\dfrac{mc^2}{mc^2+eV}\right)^2} & \text{(relativistically).} \end{cases} \tag{1}$$

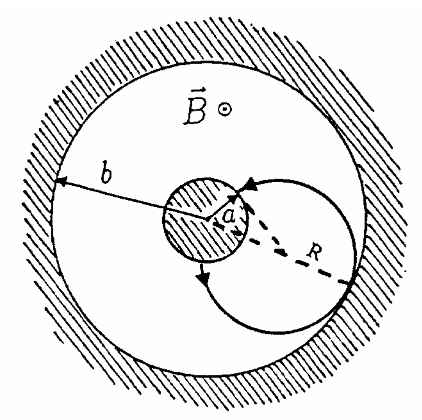
b) When $V = 0$ the electron moves in a homogeneous static magnetic field. The magnetic Lorentz force acts orthogonal to the velocity and the electron will move in a circle. The initial velocity is tangential to the circle.

The radius R of the orbit (the "cyclotron radius") is determined by equating the centripetal force and the Lorentz force:

$$eBv_0 = \frac{mv_0^2}{R},$$

i.e.

$$B = \frac{mv_0}{eR} \tag{2}$$



From the figure we see that in the critical case the radius R of the circle satisfies

$$\sqrt{a^2 + R^2} = b - R$$

By squaring we obtain

$$a^2 + R^2 = b^2 - 2bR + R^2$$

i.e.

$$R = (b^2 - a^2)/2b$$

Insertion of this value for the radius into the expression (2) gives the critical field

$$B_c = \frac{mv_0}{eR} = \underline{\underline{\frac{2bmv_0}{(b^2 - a^2)e}}}.$$

c) The change in angular momentum with time is produced by a torque. Here the azimuthal component F_φ of the Lorentz force $\vec{F} = (-e)\vec{B} \times \vec{v}$ provides a torque $F_\varphi r$. It is only the radial component $v_r = dr/dt$ of the velocity that provides an azimuthal Lorentz force. Hence

$$\underline{\underline{\frac{dL}{dt} = eBr\frac{dr}{dt}}},$$

which can be rewritten as

$$\frac{d}{dt}(L - \frac{eBr^2}{2}) = 0.$$

Hence

$$C = \underline{\underline{L - \tfrac{1}{2}eBr^2}} \qquad (3)$$

is constant during the motion. The dimensionless number k in the problem text is thus $\underline{\underline{k = \tfrac{1}{2}}}$.

d) We evaluate the constant C, equation (3), at the surface of the inner cylinder and at the maximal distance r_m:

$$0 - \tfrac{1}{2}eBa^2 = mvr_m - \tfrac{1}{2}eBr_m^2$$

which gives

$$v = \underline{\underline{\frac{eB(r_m^2 - a^2)}{2mr_m}}}. \qquad (4)$$

Alternative solution: One may first determine the electric potential $V(r)$ as function of the radial distance. In cylindrical geometry the field falls off inversely proportional to r, which requires a logarithmic potential, $V(s) = c_1 \ln r + c_2$. When the two constants are determined to yield $V(a) = 0$ and $V(b) = V$ we have

$$V(r) = V\frac{\ln(r/a)}{\ln(b/a)}.$$

The gain in potential energy, $sV(r_m)$, is converted into kinetic energy:

$$\frac{1}{2}mv^2 = eV\frac{\ln(r_m/a)}{\ln(b/a)}.$$

Thus

$$v = \sqrt{\frac{2eV}{m}\frac{\ln(r_m / a)}{\ln(b / a)}}. \qquad (5)$$

(4) and (5) seem to be different answers. This is only apparent since r_m is not independent parameter, but determined by B and V so that the two answers are identical.

e) For the critical magnetic field the maximal distance r_m equals b, the radius of the outer cylinder, and the speed at the turning point is then

$$v = \frac{eB(b^2 - a^2)}{2mb}.$$

Since the Lorentz force does not work, the corresponding kinetic energy $\frac{1}{2}mv^2$ equals eV (question a):

$$v = \sqrt{2eV/m}\ .$$

The last two equations are consistent when

$$\frac{eB(b^2 - a^2)}{2mb} = \sqrt{2eV/m}.$$

The critical magnetic field for current cut-off is therefore

$$\underline{\underline{B_c = \frac{2b}{b^2 - a^2}\sqrt{\frac{2mV}{e}}}}.$$

f) The Lorentz force has no component parallel to the magnetic field, and consequently the velocity component v_B is constant under the motion. The corresponding displacement parallel to the cylinder axis has no relevance for the question of reaching the anode.

Let v denote the final azimuthal speed of an electron that barely reaches the anode. Conservation of energy implies that

$$\tfrac{1}{2}m(v_B^2 + v_\varphi^2 + v_r^2) + eV = \tfrac{1}{2}m(v_B^2 + v^2),$$

giving

$$v = \sqrt{v_r^2 + v_\varphi^2 + 2eV/m}. \qquad (6)$$

Evaluating the constant C in (3) at both cylinder surfaces for the critical situation we have

$$mv_\varphi a - \tfrac{1}{2}eB_c a^2 = mvb - \tfrac{1}{2}eB_c b^2.$$

Insertion of the value (6) for the velocity v yields the critical field

$$B_c = \frac{2m(vb - v_\varphi a)}{e(b^2 - a^2)} = \frac{2mb}{e(b^2 - a^2)}\left[\sqrt{v_r^2 + v_\varphi^2 + 2eV/m} - v_\varphi\, a/b\right].$$

Solution Problem 3

a) With the centre of the earth as origin, let the centre of mass C be located at $\vec{l}$. The distance l is determined by

$$M\,l = M_m\,(L - l),$$

which gives

$$l = \frac{M_m}{M + M_m} L = \underline{\underline{4.63 \cdot 10^6\,\text{m},}} \qquad (1)$$

less than R, and thus inside the earth.

The centrifugal force must balance the gravitational attraction between the moon and the earth:

$$M\omega^2 l = G\frac{MM_m}{L^2},$$

which gives

$$\omega = \sqrt{\frac{GM_m}{L^2 l}} = \underline{\underline{\sqrt{\frac{G(M + M_m)}{L^3}}}} = \underline{\underline{2.67 \cdot 10^{-6}\,\text{s}^{-1}}}. \qquad (2)$$

(This corresponds to a period $2\pi/\omega = 27.2$ days.) We have used (1) to eliminate l.

b) The potential energy of the mass point m consists of three contributions:

(1) Potential energy because of rotation (in the rotating frame of reference, see the problem text),

$$-\frac{1}{2}m\omega^2 r_1^2,$$

where $\vec{r}_1$ is the distance from C. This corresponds to the centrifugal force $m\omega^2 r_1$, directed outwards from C.

(2) Gravitational attraction to the earth,

$$-G\frac{mM}{r}.$$

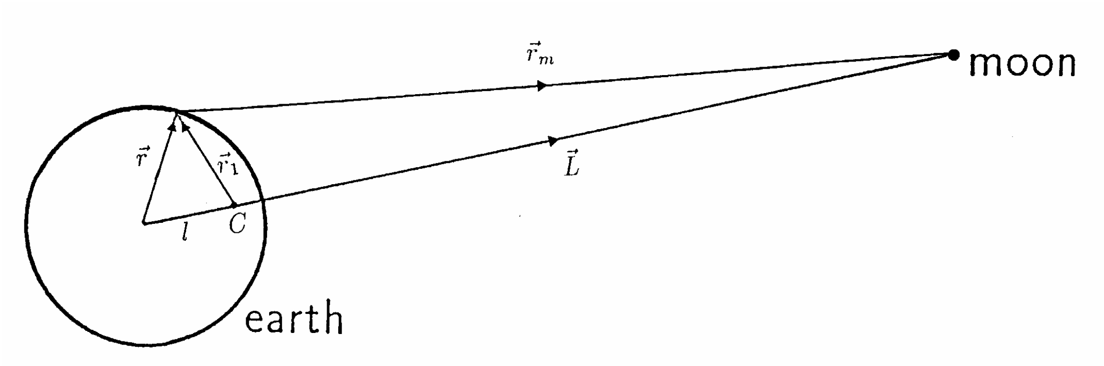
(3) Gravitational attraction to the moon,

$$-G\frac{mM_m}{r_m},$$

where $\vec{r}_m$ is the distance from the moon.

Describing the position of m by polar coordinates $r,\ \varphi$ in the plane orthogonal to the axis of rotation (see figure), we have

$$\vec{r}_1^{\,2} = (\vec{r} - \vec{l}\,)^2 = r^2 - 2rl\cos\varphi + l^2.$$



Adding the three potential energy contributions, we obtain

$$V(\vec{r}) = -\frac{1}{2}m\omega^2(r^2 - 2rl\cos\varphi + l^2) - G\frac{mM}{r} - G\frac{mM_m}{\left|\vec{L} - \vec{r}\right|}. \tag{3}$$

Here l is given by (1) and

$$\left|\vec{r}_m\right| = \sqrt{(\vec{L} - \vec{r})^2} = \sqrt{L^2 - 2\vec{L}\vec{r} + r^2} = L\sqrt{1 + (r/L)^2 - 2(r/L)\cos\varphi},$$

c) Since the ratio $r/L = a$ is very small, we may use the expansion

$$\frac{1}{\sqrt{1 + a^2 - 2a\cos\varphi}} = 1 + a\cos\varphi + a^2\,\tfrac{1}{2}(3\cos^2\varphi - 1).$$

Insertion into the expression (3) for the potential energy gives

$$V(r,\varphi)/m = -\tfrac{1}{2}\omega^2 r^2 - \frac{GM}{r} - \frac{GM_m r^2}{2L^3}(3\cos^2\varphi - 1), \tag{4}$$

apart from a constant. We have used that

$$m\omega^2 rl\cos\varphi - GmM_m\frac{r}{L^2}\cos\varphi = 0,$$

when the value of ω^2, equation (2), is inserted.

The form of the liquid surface is such that a mass point has the same energy V *everywhere on the surface*. (This is equivalent to requiring no net force tangential to the surface.) Putting

$$r = R + h,$$

where the tide h is much smaller than R, we have approximately

$$\frac{1}{r}=\frac{1}{R+h}=\frac{1}{R}\frac{1}{1+(h/R)}\cong\frac{1}{R}(1-\frac{h}{R})=\frac{1}{R}-\frac{h}{R^2},$$

as well as

$$r^2 = R^2 + 2Rh + h^2 \cong R^2 + 2Rh.$$

Inserting this, and the value (2) of ω into (4), we have

$$V(r,\varphi)/m = -\frac{G(M+M_m)R}{L^3}h + \frac{GM}{R^2}h - \frac{GM_m r^2}{2L^3}(3\cos^2\varphi - 1), \qquad (5)$$

again apart from a constant.

The magnitude of the first term on the right-hand side of (5) is a factor

$$\frac{(M+M_m)}{M}(\frac{R}{L})^3 \cong 10^{-5}$$

smaller than the second term, thus negligible. If the remaining two terms in equation (5) compensate each other, *i.e.*

$$h = \underline{\frac{M_m r^2 R^2}{2ML^3}(3\cos^2\varphi - 1)},$$

then the mass point *m* has the same energy everywhere on the surface. Here r^2 can safely be approximated by R^2, giving the tidal bulge

$$h = \frac{M_m R^4}{2ML^3}(3\cos^2\varphi - 1).$$

The largest value $h_{\max} = M_m R^4 / ML^3$ occurs for $\varphi = 0$ or π, in the direction of the moon or in the opposite direction, while the smallest value $h_{\min} = -M_m R^4 / 2ML^3$ corresponds to $\varphi = \pi/2$ or $3\pi/2$. The difference between high tide and low tide is therefore

$$h_{\max} - h_{\min} = \underline{\underline{\frac{3M_m R^4}{2ML^3}}} = \underline{\underline{0.54\text{m}.}}$$

(The values for high and low tide are determined up to an additive constant, but the difference is of course independent of this.)