XXVI International Physics Olympiad

Canberra, ACT
Australia

Theoretical Competition

July 7, 1995

Time Allowed: 5 Hours

READ THIS FIRST

Permitted Materials: Non Programable Calculators

Instructions:

1. **Use only the pen provided**

2. **Use only the marked side of the paper**

3. **Begin each problem on a separate sheet**

4. **Write at the top of every sheet:**

   - **The number of the problem**
   - **The number of the sheet in your solution for each problem**
   - **The total number of sheets in your solution to the problem.**

Example (for Problem 1 with 3 sheets): 1 1/3
1 2/3
1 3/3

Do not staple your sheets. They will be clipped together for you at the end of the examination.

Question 1

Gravitational Red Shift and the Measurement of Stellar Mass

(a) (3 marks)
A photon of frequency f possesses an effective inertial mass m determined by its energy. We may assume that it has a gravitational mass equal to this inertial mass. Accordingly, a photon emitted at the surface of a star will lose energy when it escapes from the star's gravitational field. Show that the frequency shift Δf of a photon when it escapes from the surface of the star to infinity is given by

$$\frac{\Delta f}{f} \simeq -\frac{GM}{Rc^2}$$

for $\Delta f << f$ where
G = gravitational constant
R = radius of the star
c = velocity of light
M = mass of the star.

Thus, the red shift of a known spectral line measured a long way from the star can be used to measure the ratio M/R. Knowledge of R will allow the mass of the star to be determined.

(b) (12 marks)
An unmanned spacecraft is launched in an experiment to measure both the mass M and radius R of a star in our galaxy. As the spacecraft approaches its objective radially, photons emitted from He^+ ions on the surface of the star are monitored via resonance excitation of a beam of He^+ ions in a test chamber inside the spacecraft. Resonance absorption occurs only if the He^+ ions are given a velocity towards the star to allow exactly for the red shifts. The velocity ($v = \beta c$) of the He^+ ions in the spacecraft relative to the star at absorption resonance is measured as a function of the distance d from the (nearest) surface of the star. The experimental data are displayed in the accompanying table. Fully utilize the data to determine graphically the mass M and radius R of the star. There is no need to estimate the uncertainties in your answer.

Data for Resonance Condition

Velocity parameter	$\beta = v/c$ (10^{-5})	3.352	3.279	3.195	3.077	2.955
Distance from surface of star	d (10^{8}m)	38.90	19.98	13.32	8.99	6.67

(c) In order to determine R and M in such an experiment, it is usual to consider the frequency correction due to the recoil of the emitting atom. [Thermal motion causes emission lines to be broadened without displacing distribution maxima, and we may therefore assume that all thermal effects have been taken into account.]

(i) (4 marks)
Let E be the energy difference between two atomic energy levels, with the atom at rest in each case. Assume that the atom decays at rest, producing a photon and a recoiling atom. Obtain the relativistic expression for the energy hf of a photon emitted in terms of E and the initial rest mass m_o of the atom.

(ii) (1 mark)

Hence, make a numerical estimate of the relativistic frequency shift $\left(\frac{\Delta f}{f}\right)_{\text{recoil}}$ for the case of He^+ ions.

Your answer should turn out to be much smaller than the gravitational red shift obtained in part (b).

Data:

Velocity of light	c	=	3.0×10^8 m/s
Rest energy of He	$m_o c^2$	=	4×938 (MeV)
Bohr energy	E_n	=	$-\frac{13.6\ Z^2}{n^2}$ (eV)
Gravitational constant	G	=	6.7×10^{-11} Nm^2 kg^{-2}.

Question 2

Sound Propagation

Introduction

The speed of propagation of sound in the ocean varies with depth, temperature and salinity. Figure 1(a) below shows the variation of sound speed c with depth z for a case where a minimum speed value c_o occurs midway between the ocean surface and the sea bed. Note that for convenience z = 0 at the depth of this sound speed minimum, $z = z_s$ at the surface and $z = -z_b$ at the sea bed. Above z = 0, c is given by;

$$c = c_o + bz$$

Below z = 0, c is given by;

$$c = c_o - bz$$

In each case $b = \left|\frac{dc}{dz}\right|$, that is, b is the magnitude of the sound speed gradient with depth; b is assumed constant.

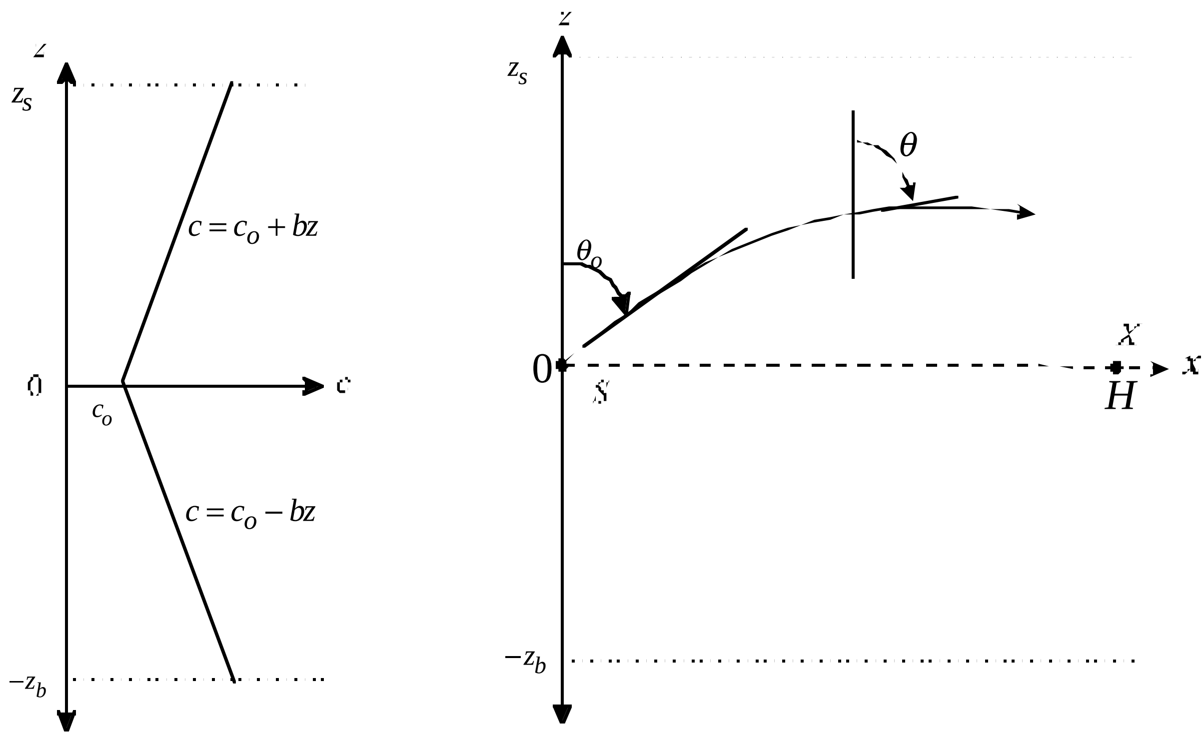


Figure 1 (a) **Figure 1 (b)**

Figure 1(b) shows a section of the *z*-*x* plane through the ocean, where *x* is a horizontal direction. At all points along the *z*-*x* section the sound speed profile $c(z)$ is as shown in figure 1(a). At the position $z = 0$, $x = 0$, a sound source *S* is located. Part of the output from this source is described by a sound ray emerging from *S* with initial angle θ_o as shown. Because of the variation of sound speed with *z*, the ray will be refracted, leading to varying values along the trajectory of the ray.

(a) (6 marks)
Show that the initial trajectory of the ray leaving the source *S* and constrained to the *z*-*x* plane is an arc of a circle with radius *R* where:

$$R = \frac{c_o}{b \sin \theta_o} \quad \text{for} \quad 0 \le \theta_0 < \frac{\pi}{2}$$

(b) (3 marks)
Derive an expression involving z_S, c_O and *b* to give the smallest value of the angle θ_o for upwardly directed rays which can be transmitted without the sound wave reflecting from the sea surface.

(c) (4 marks)
Figure 1(b) shows the position of a sound receiver *H* which is located at the position $z = 0$, $x = X$. Derive an expression involving *b*, *X* and c_O to give the series of values of angle θ_o required for the sound ray emerging from *S* to reach the receiver *H*. Assume that z_S and z_b are sufficiently large to remove the possibility of reflection from sea surface or sea bed.

(d) (2 marks)
Calculate the smallest four values of θ_o for refracted rays from *S* to reach *H* when;

$X = 10{,}000$ m
$c_O = 1{,}500$ m/s
$b = 0.02000\text{s}^{-1}$

(e) (5 marks)
Derive an expression to give the time taken for sound to travel from *S* to *H* following the ray path associated with the **smallest** value of angle θ_o, as determined in part (c). Calculate the value of this transit time for the conditions given in part (d). The following result may be of assistance:

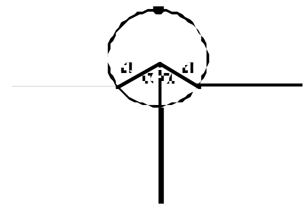
$$\int \frac{dx}{\sin x} = \ln \tan\left(\frac{x}{2}\right)$$

Calculate the time taken for the direct ray to travel from *S* to *H* along $z = 0$. Which of the two rays will arrive first, the ray for which $\theta_o = \frac{\pi}{2}$, or the ray with the smallest value of θ_o as calculated for part (d)?

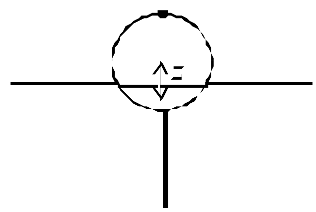
Question 3

Cylindrical Buoy

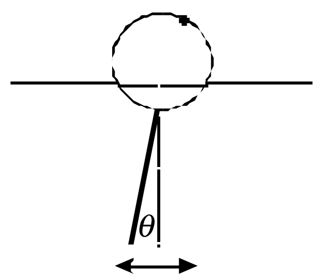
(a) (3 marks)
A buoy consists of a solid cylinder, radius a, length l, made of lightweight material of uniform density d, with a uniform rigid rod protruding directly outwards from the bottom halfway along the length. The mass of the rod is equal to that of the cylinder, its length is the same as the diameter of the cylinder and the density of the rod is greater than that of seawater. This buoy is floating in seawater (of density ρ). In equilibrium derive an expression relating the floating angle α, as drawn, to d/ρ. Neglect the volume of the rod.



(b) (4 marks)
If the buoy, due to some perturbation, is depressed vertically by a small amount z, it will experience a net upthrust, which will cause it to begin oscillating up and down, about the equilibrium floating position. Determine the frequency of this vertical mode of vibration in terms of α, g and a, where g is the acceleration due to gravity, assuming the influence of water motion on the dynamics of the buoy is such as to increase the effective mass of the buoy by a factor of one third. You may assume that α is not small.



(c) (8 marks)
In the approximation that the cylinder swings about it's horizontal central axis, determine the frequency of swing again in terms of g and a. Neglect the dynamics and viscosity of the water in this case. The angle of swing is supposed to be small.



(d) (5 marks)
The buoy contains sensitive accelerometers which can measure the vertical and swinging motions and can relay this information by radio to shore. In relatively calm waters it is recorded that the vertical oscillation period is about 1 second and the swinging oscillation period is about 1.5 seconds. From this information, show that the float angle is about 90° and thereby estimate the radius of the buoy and its total mass, given that the cylinder length l equals a.

[You may take it that $\rho \simeq 1000\ \mathrm{Kg\ m^{-3}}$ and $g \simeq 9.8\ \mathrm{m\ s^{-2}}$.]